A Geometric Analysis of Global Profit Maximization for a Two-Product Firm

Stephen K. Layson[1]

ABSTRACT

This paper analyzes several fundamental issues that arise in verifying a global maximum for a *seemingly* simple economic problem, profit maximization for a two-product firm. A new gradient path geometrical method of verifying a global maximum is presented that is analogous to the use of phase diagrams to solve for the equilibrium of a first-order differential equation system in two variables. An important advantage of the geometrical approach is that it always verifies a global profit maximum when the profit function is concave and it also verifies a global maximum in many cases where the profit function is neither concave nor quasiconcave.

Introduction

Often in economics the distinction between a local and global maximum is ignored.[2] The reason for this is the implicit assumption that economic objective functions typically have a single local maximum which is also the global maximum.[3] Whether this assumption is generally valid or not, the theory of a global maximum is one of the cornerstones on which economics rests so we should be clear about this distinction. In Economics two basic assumptions are that firms are attempting to find the *global* maximum of profit and consumers are attempting to find the *global* maximum of utility. Simon and Blume (1994, p. 518), the authors of *the* leading text in mathematical economics, explain well the importance of global maxima and the closely linked concept of concave functions in economics:

> The property that critical points of concave functions are global maximizers is an important one in economic theory. For example, many economic principles, such as marginal rates of substitution equals the price ratio, or marginal revenue equals marginal cost are simply the first order *necessary* conditions of the corresponding maximization problem. Ideally an economist would like such a rule to also be a *sufficient* condition guaranteeing that utility or profit is maximized so that it can provide a guideline for economic behavior. This situation does occur when the objective function is concave. Furthermore, an economist, who wants to analyze how the maximizer in a parameterized problem depends on the parameters involved, will usually apply the implicit function theorem to the equations of the first order necessary conditions for maximization. The only situation in which it can be guaranteed that the solution to these perturbed equations is indeed a maximum for all values of the parameters occurs when the objective function is concave.

[1] Stephen K. Layson, Department of Economics, 457 Bryan Building, UNCG, Greensboro, NC 27412-5001 USA (336)-334-4868 layson@uncg.edu. I wish to thank M. Taylor Rhodes for his expert assistance with the figures in this paper as well as editorial help.

[2] For example, the possibility that there could be a difference between the global and local maximum in an economic application is never mentioned in Samuelson (1972) or Silberberg and Suen (2001).

[3] For a simple yet important example of an economic maximization problem with two local maxima see Battalio and Ekelund (1972).

As pointed out in the preceding quotation[4], in economics the most common procedure for justifying an unconstrained global maximum is to either establish or assume that the objective function is differentiable and concave, so that the satisfaction of the first order conditions guarantees a global maximum. Note that the rather strong assumption of concavity can be relaxed some. For example, Ponstein (1967)[5] proved that a local maximum of a strictly quasiconcave function is also a global maximum. Also, for objective functions with closed and bounded domains, we can often use Weierstrass's theorem to prove the existence of both a global maximum and a global minimum.[6] Sydsaeter and Hammond (1995, pp. 608-611) give a nice treatment of how to find the global maximum and the global minimum for this case.

This paper analyzes several fundamental issues that arise in verifying a global maximum for a seemingly simple economic problem, profit maximization for a two-product firm. Section II of the paper gives several plausible examples of profit functions that are neither concave nor quasiconcave. Section III of the paper presents a new gradient path geometrical method of verifying a global maximum that is analogous to the use of phase diagrams to solve for the equilibrium of a first-order differential equation system in two variables.[7] An important advantage of the geometrical approach to optimization problems is that it always verifies a global maximum when the objective function is concave and it also verifies a global maximum in many cases where the objective function is neither concave nor quasiconcave.

Concavity, Quasiconcavity and the Firm Profit Function

Consider a two-product firm's profit function given by the C^2 function, $\pi(q_1,q_2)$ where q_1 the output of product 1 is, q_2 is the output of product 2, $R(q_1,q_2)$ is the firm's revenue function and $C(q_1,q_2)$ is the firm's cost function. Equation (1) expresses the profit function as revenue minus cost.

$$(1) \qquad \pi(q_1,q_2) = R(q_1,q_2) - C(q_1,q_2)$$

The domain of the profit function is all nonnegative values of q_1 and q_2 with no upper bounds placed on either q_1 or q_2. Because profit is revenue minus cost, the profit function is necessarily concave if the revenue function is concave and the cost function is convex. Thus, there are two ways in which the profit function fails to be concave, either the revenue function is not concave or the cost function is not convex. This section will demonstrate that the assumption that the profit function is concave is really very stringent.

Let $\pi_i = \frac{\partial \pi}{\partial q_i}$ denote the first order partial derivatives of the profit function and let $\pi_{ij} = \frac{\partial \pi_j}{\partial q_i}$ denote the second order partial derivatives of the profit function. The well known necessary conditions for concavity of the profit function is that at all points in the domain $\pi_{11} \le 0$, $\pi_{22} \le 0$ and $\pi_{11}\pi_{22} \ge \pi_{12}^2$. The most common type of violation of the concavity assumption occurs when either π_{11} or π_{22} are positive at some point. Even when π_{11} and π_{22} are everywhere negative, however, violations of the other condition for concavity, $\pi_{11}\pi_{22} \ge \pi_{12}^2$ are also quite possible.

[4] The last sentence in the preceding quotation needs to be qualified. A function does not necessarily have to be concave for the satisfaction of the first order conditions to guarantee a maximum for all values of a parameter. Consider the function $f(x;a) = e^{-x^2+ax}$. This function is not concave yet the maximizing value of *x* is *a/2* for all values of the parameter *a*. While *f* is not concave in this example it is a monotonic increasing transformation of a concave function.

[5] Also see Mangasarian (1965) and Martos (1965).

[6] See for example Simon and Blume (1994, Theorem 30.1, p.823).

[7] See Layson (2001) for an earlier discussion of the gradient path concept and Chiang and Wainwright (2005, pp. 614-623) for a good discussion of two-variable phase diagrams.

Equation (2) expresses the second derivatives of the profit function in terms of the second derivatives of the revenue and cost functions

$$(2) \qquad \pi_{ii}(q_1,q_2) = R_{ii}(q_1,q_2) - C_{ii}(q_1,q_2)\,.$$

A sufficient condition for π_{ii} to be positive is that marginal cost fall more sharply than marginal revenue. For example, for a two-product firm that sells both products in perfectly competitive markets so that $R_{11} = R_{22} = 0$, falling marginal cost of producing either product $(C_{ii} < 0)$ at any point in the domain will violate the concavity assumption. For a two-product monopoly the possibility of π_{ii} >0 can arise either by rising marginal revenue ($R_{ii} > 0$) and/or falling marginal cost $(C_{ii} < 0)$.[8]

Positive values for π_{11} and/or π_{22} can also easily violate the weaker assumption of quasiconcavity. The necessary condition for quasiconcavity is that $-\pi_1^2\pi_{22} + 2\pi_1\pi_2\pi_{12} - \pi_2^2\pi_{11} \geq 0$ at all points in the domain. In the case where $\pi_{12} = 0$ at all points in the domain the necessary condition for quasiconcavity simplifies to $-\pi_1^2\pi_{22} - \pi_2^2\pi_{11} \geq 0$ which is clearly violated (at non-critical points) when π_{11} and π_{22} are both positive.

The following 3 examples explicitly demonstrate different ways in which the profit function can fail to be either concave or quasiconcave.

Example One

This example illustrates a case where the profit function fails to be either concave or quasiconcave because of falling marginal cost. A price-taking, two-product firm has a cubic cost function given by

$$(3) \qquad C(q_1,q_2) = F + \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}\beta_1 q_1^2 - \frac{1}{2}\beta_2 q_2^2 + \frac{1}{3}\gamma_1 q_1^3 + \frac{1}{3}\gamma_2 q_2^3\,.$$

It is assumed that the parameters $F, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 are all positive. The marginal cost of producing each product is given by

$$(4) \qquad C_i = \alpha_i - \beta_i q_i + \gamma_i q_i^2 \qquad i = 1,2\,.$$

To insure that the marginal cost of producing each commodity is always positive it is assumed that $2\sqrt{\alpha_1\gamma_1} > \beta_1$ and $2\sqrt{\alpha_2\gamma_2} > \beta_2$.

The second derivatives of this cost function are

$$(5) \qquad C_{ii} = -\beta_i + 2\gamma_i q_i \qquad i = 1,2$$

---

[8] See Formby, Layson, and Smith (1982) for a discussion of the possibility of upward sloping marginal revenue.

(6) $C_{ij} = 0 \qquad i \neq j$.

For $q_i = 0$, $C_{ii} = -\beta_i < 0$. Because this is a price-taking firm it follows from equations (2) and (5) that at $q_i = 0$, $\pi_{ii} = -C_{ii} = \beta_i > 0$. Thus the profit function is not concave. Because $\pi_{12} = 0$ in this example, it follows that at $q_1 = q_2 = 0$ the necessary condition for quasiconcavity, $-\pi_1^2\pi_{22} + 2\pi_1\pi_2\pi_{12} - \pi_2^2\pi_{11} \geq 0$ is also violated.

Example Two

In this example the profit function is not concave nor quasiconcave even though $\pi_{11} \leq 0$ and $\pi_{22} \leq 0$ at all points in the domain. A price-taking, two-product firm's cost function is

(7) $$C(q_1, q_2) = F + a_1q_1 + a_2q_2 + b_1q_1^{n_1} + b_2q_2^{n_2} + hq_1q_2 .$$

In the cost function above a_i , b_i , n_i and h are parameters with restrictions $F \geq 0$, $a_i \geq 0$, $b_i > 0$ and $n_i > 1$. As will be demonstrated below, for the cost function given by equation (7), for $h \neq 0$ a necessary condition for convexity of the cost function (concavity of the profit function) is that $n_1 = n_2 = 2$. The second derivatives of the cost function are:

(8) $$C_{11} = (n_1 - 1)n_1b_1q_1^{n_1-2} \geq 0$$

(9) $$C_{22} = (n_2 - 1)n_2b_2q_2^{n_2-2} \geq 0$$

(10) $$C_{12} = h$$

Because in this example $R_{11} = R_{22} = 0$, the convexity of the cost function is equivalent to concavity of the profit function. The inequalities $\pi_{11} = -C_{11} \leq 0$ and $\pi_{22} = -C_{22} \leq 0$ are satisfied at all points in the domain in this example, but if $h \neq 0$ the inequality, $\pi_{11}\pi_{22} \geq \pi_{12}^2$ can *only* be satisfied at all points in the domain when $n_1 = n_2 = 2$. If either n_1 or n_2 is not exactly equal to *2* then the profit function is not concave. To see this use equations (8)-(10) to rewrite the necessary condition $\pi_{11}\pi_{22} \geq \pi_{12}^2$ as

(11) $$(n_1 - 1)(n_2 - 1)n_1n_2b_1b_2q_1^{n_1-2}q_2^{n_2-2} \geq h^2$$

First note that for $n_1 = n_2 = 2$ inequality (11) simplifies to $4b_1b_2 \geq h^2$ and as long as the parameters satisfy this latter inequality the profit function is concave.[9] For $n_1 > 2$, note that $q_1^{n_1-2}$, which is part of the left-hand-side of inequality (11), goes to zero as q_1 goes to zero. Hence for $n_1 > 2$, inequality (11) fails to hold

[9] Chiang and Wainwright (2005, 331-32) discuss an optimization problem of this type in their classic text.

at all points in the domain. For $n_1 < 2$, $q_1^{n_1-2}$ goes to zero as q_1 goes to infinity. Hence for $n_1 < 2$, inequality (11) also fails to hold at all points in the domain. It follows that the profit function is not concave for either $n_1 > 2$ or $n_1 < 2$. Similar reasoning will verify that the profit function is not concave for either $n_2 > 2$ or $n_2 < 2$.

It is easy to demonstrate that this profit function can also fail to be quasiconcave. For one example, (many more can be found) consider the case where: $n_1 > 2$, $n_2 > 2$ and $q_1 = q_2 = 0$. In this case $\pi_{11} = \pi_{22} = 0$ and the necessary condition for quasiconcavity, $-\pi_1^2\pi_{22} + 2\pi_1\pi_2\pi_{12} - \pi_2^2\pi_{11} \geq 0$ simplifies to $-2(p_1 - a_1)(p_2 - a_2)h \geq 0$. Obviously this latter inequality fails to hold if, for example, $p_1 > a_1, p_2 > a_2$ and *h>0*.

Example Three

In examples 1 and 2 the profit function failed to be concave because the cost functions were not convex. The next example illustrates a case where the profit function generally fails to be concave because the revenue function is only concave in a special case. Consider a two-product monopoly with a linear, convex, cost function $C(q_1, q_2) = F + a_1q_1 + a_2q_2$ that faces inverse demand functions given by:

$$(12) \qquad p_1 = \alpha_1 - \beta_1 q_1^{n_1} - \gamma_1 q_2$$

$$(13) \qquad p_2 = \alpha_2 - \gamma_2 q_1 - \beta_2 q_2^{n_2}.$$

Note that for the special case where $n_1 = n_2 = 1$ the inverse demand functions above are linear. It is assumed in equations (12) and (13) that $\alpha_1, \alpha_2, \beta_1, \beta_2, n_1, n_2 > 0$. The signs of γ_1 and γ_2 may be either positive or negative depending on whether the two goods are substitutes or complements. The revenue function in this example is

$$(14) \qquad R(q_1, q_2) = \alpha_1 q_1 + \alpha_2 q_2 - (\gamma_1 + \gamma_2)q_1 q_2 - \beta_1 q_1^{n_1+1} - \beta_2 q_2^{n_2+1}.$$

Because $C_{11} = C_{22} = 0$ in this example, the second derivatives of the profit function are:

$$(15) \qquad \pi_{11} = R_{11} = -(n_1 + 1)n_1\beta_1 q_1^{n_1-1} \leq 0$$

$$(16) \qquad \pi_{22} = R_{22} = -(n_2 + 1)n_2\beta_2 q_2^{n_2-1} \leq 0$$

$$(17) \qquad \pi_{12} = R_{12} = -(\gamma_1 + \gamma_2)$$

The inequalities $\pi_{11} \leq 0$ and $\pi_{22} \leq 0$ are satisfied at all points in the domain in this example, but if $\gamma_1 + \gamma_2 \neq 0$, the inequality, $\pi_{11}\pi_{22} \geq \pi_{12}^2$ can *only* be satisfied at all points in the domain when $n_1 = n_2 = 1$ and $4\beta_1\beta_2 \geq (\gamma_1 + \gamma_2)^2$. If $\gamma_1 + \gamma_2 \neq 0$ and either n_1 or n_2 is not exactly equal to 1 then the profit function is not concave. To see this use equations (15)-(17) to rewrite the necessary condition $\pi_{11}\pi_{22} \geq \pi_{12}^2$ as

$$(18)\qquad (n_1+1)(n_2+1)n_1n_2\beta_1\beta_2 q_1^{n_1-1}q_2^{n_2-1} \geq (\gamma_1+\gamma_2)^2 .$$

First note that for $n_1 = n_2 = 1$ inequality (18) simplifies to $4\beta_1\beta_2 \geq (\gamma_1+\gamma_2)^2$ and as long as the parameters satisfy this latter inequality the profit function is concave. For $n_1 > 1$, note that $q_1^{n_1-1}$, which is part of the left-hand-side of inequality (18), goes to zero as q_1 goes to zero. Hence for $n_1 > 1$, inequality (18) fails to hold at all points in the domain. For $n_1 < 1$, $q_1^{n_1-1}$ goes to zero as q_1 goes to infinity. Hence for $n_1 < 1$, inequality (18) also fails to hold at all points in the domain. It follows that the profit function is not concave for either $n_1 > 1$ or $n_1 < 1$. Similar reasoning will verify that the profit function is not concave for either $n_2 > 1$ or $n_2 < 1$.

It is easy to demonstrate that this profit function can also fail to be quasiconcave. For one example, consider the case where $n_1 > 1$, $n_2 > 1$ and $q_1 = q_2 = 0$. In this case $\pi_{11} = \pi_{22} = 0$ and the necessary condition for quasiconcavity, $-\pi_1^2\pi_{22} + 2\pi_1\pi_2\pi_{12} - \pi_2^2\pi_{11} \geq 0$ simplifies to $-2a_1a_2(\gamma_1+\gamma_2) \geq 0$. Obviously this latter inequality fails to hold if $\gamma_1+\gamma_2 > 0$.

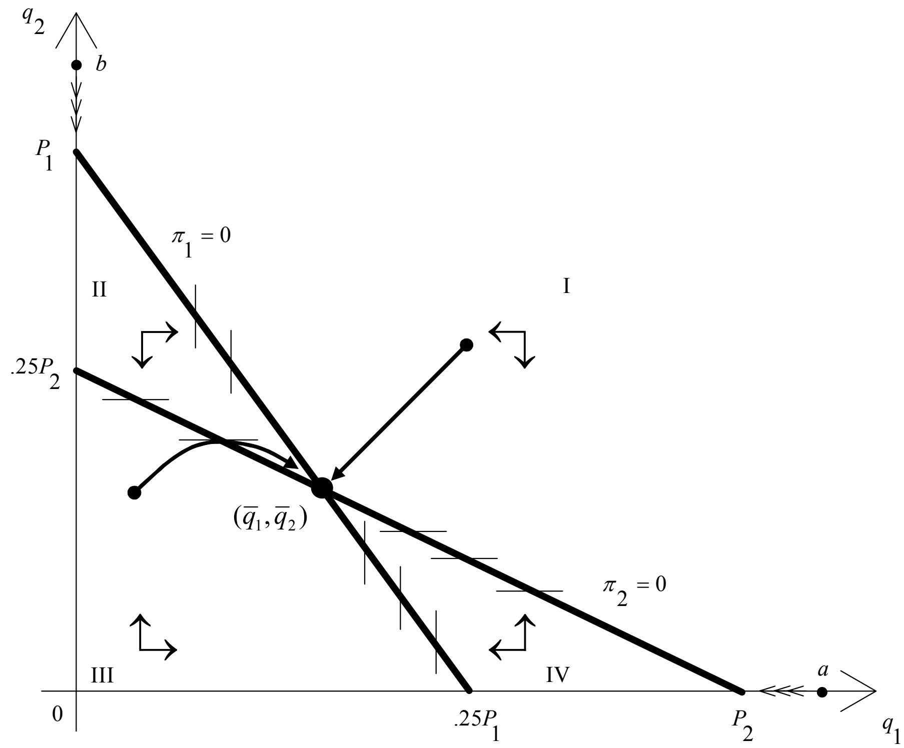
The Geometrical Approach for Verifying a Global Profit Maximum

As the last section has demonstrated, it is easy to construct examples of plausible profit functions that are neither concave nor quasiconcave. This section presents a geometrical method of verifying a global maximum that does not require that the objective function be either concave or quasiconcave. To illustrate the geometrical approach consider the simple concave two-product profit function discussed in Chiang and Wainwright's (2005, 331-32) classic text

$$(19)\qquad \pi(q_1,q_2) = p_1q_1 + p_2q_2 - 2q_1^2 - q_1q_2 - 2q_2^2 .$$

The first order conditions for maximizing profit are $\pi_1 = p_1 - 4q_1 - q_2 = 0$ and $\pi_2 = p_2 - q_1 - 4q_2 = 0$. Figure 1 shows the graphs of these two first order conditions in the (q_1,q_2) plane. The single intersection of the linear first order conditions is the unique critical point of the profit function and is labeled $(\overline{q}_1,\overline{q}_2)$ in figure 1. The intersecting first order conditions in figure 1 divide the domain into 4 regions labeled I through IV. The rightward pointing arrows denote regions in which $\pi_1 > 0$ and the leftward pointing arrows denote regions where $\pi_1 < 0$. The upward pointing arrows denote regions in which $\pi_2 > 0$ and the downward pointing arrows denote regions in which $\pi_2 < 0$.

Figure 1



In region I of figure 1, the gradients all point in a southwest direction, in region II the gradients all point in a southeast direction, in region III the gradients all point in a northeast direction and in region IV the gradients all point in a northwest direction. Starting at any arbitrary point in the domain other than the critical point, $(\bar{q}_1, \bar{q}_2)$, the gradients will point towards higher values of π. Figure 1 illustrates a few gradient paths which are analogous to phase paths in two-variable phase diagrams. As one moves along a path in the direction indicated by the gradients the value of π steadily rises. If all non-critical points in the domain have gradient paths leading to the same point, this end point is the global maximum.

It is important to note that when a gradient path crosses the $\pi_1 = 0$ locus, the gradient path is vertical at this crossing point and when a gradient path crosses the $\pi_2 = 0$ locus, the gradient path is horizontal at this crossing point. Therefore, if one starts at any point in either region II or IV of figure 1, the gradient path must lead to $(\bar{q}_1, \bar{q}_2)$ without passing into regions I or II. If one starts at a point in region I however, there are 3 possibilities: (1) the gradient path may approach $(\bar{q}_1, \bar{q}_2)$ without crossing into either region II or IV, (2) the gradient path may cross from region I into region II where it stays until it reaches $(\bar{q}_1, \bar{q}_2)$ or (3) the gradient path may cross from region I to region IV where it stays until it reaches $(\bar{q}_1, \bar{q}_2)$. If one starts at a point in region I on the q_1 axis such as point *"a"* in figure 1 the unrestricted gradient will point in the southwest direction. Because of the non-negativity restriction on q_2 in this problem the gradient path travels leftward along the q_1 axis until it crosses into region IV where it then approaches $(\bar{q}_1, \bar{q}_2)$. Similarly, if one starts at a point in region I on the q_2 axis such as point *"b"* in figure 1 the unrestricted gradient will point in the southwest direction. Because of the non-negativity restriction on q_1 when $q_1 = 0$ in figure 1 the gradient path travels downward along the q_2 axis until it crosses into region II where it then approaches $(\bar{q}_1, \bar{q}_2)$.

If one starts at any point in region III, there are 3 possibilities: (1) the gradient path may approach $(\bar{q}_1,\bar{q}_2)$ without crossing into regions II or IV, (2) the gradient path may cross from region III into region II where it stays until it reaches $(\bar{q}_1,\bar{q}_2)$ or (3) the gradient path crosses from region III into region IV where it stays until it reaches $(\bar{q}_1,\bar{q}_2)$. Because all possible non-critical points in the domain have gradient paths that lead to the point $(\bar{q}_1,\bar{q}_2)$ this point is the global maximizer of π .[10]

Note in figure 1 that the $\pi_1 = 0$ locus is steeper than the $\pi_2 = 0$ locus at the critical point $(\bar{q}_1,\bar{q}_2)$ .This condition, along with the condition that $\pi_{11},\pi_{22} < 0$ are equivalent to the second order sufficient conditions for a *local* maximum. Applying the implicit function rule of differentiation to the first order conditions, $\pi_1(q_1,q_2) = 0$ and $\pi_2(q_1,q_2) = 0$ the point slopes of the two loci are, respectively:

$$(20) \quad \frac{dq_2}{dq_1}\Big|_{\pi_1=0} = -\frac{\pi_{11}}{\pi_{12}} \quad (\pi_{12} \neq 0)$$

$$(21) \quad \frac{dq_2}{dq_1}\Big|_{\pi_2=0} = -\frac{\pi_{12}}{\pi_{22}} \quad (\pi_{22} \neq 0).$$

Recall that the second order sufficient conditions for a *local* maximum are that the second derivatives evaluated at a critical point satisfy $\pi_{11},\pi_{22} < 0$ and $\pi_{11}\pi_{22} > \pi_{12}^2$. *If at a critical point* $\pi_{11},\pi_{22} < 0$ *is satisfied then the critical point will be a local maximizer if the* $\pi_1 = 0$ *locus is steeper than the* $\pi_2 = 0$ *locus.* To demonstrate this, consider the 3 possible sub-cases where $\pi_{11},\pi_{22} < 0$: (1) $\pi_{12} > 0$, (2) $\pi_{12} < 0$ and (3) $\pi_{12} = 0$. From equations (20) and (21) observe that if $\pi_{12} > 0$ then both loci have positive slopes and if the $\pi_1 = 0$ locus is steeper than the $\pi_2 = 0$ locus this is equivalent to $-\frac{\pi_{11}}{\pi_{12}} > -\frac{\pi_{12}}{\pi_{22}}$ which implies $\pi_{11}\pi_{22} > \pi_{12}^2$. If $\pi_{12} < 0$ then both loci have negative slopes and if the $\pi_1 = 0$ locus is steeper than the $\pi_2 = 0$ locus this is equivalent to $\frac{\pi_{11}}{\pi_{12}} > \frac{\pi_{12}}{\pi_{22}}$ which implies that $\pi_{11}\pi_{22} > \pi_{12}^2$. Finally, if $\pi_{12} = 0$ then the $\pi_1 = 0$ locus is vertical and the $\pi_2 = 0$ locus is horizontal so the $\pi_1 = 0$ locus is clearly steeper than the $\pi_2 = 0$ locus, and the condition $\pi_{11}\pi_{22} > \pi_{12}^2$ is satisfied.

The profit function given by equation (19) is strictly concave and the intersection of the first order conditions in figure 1 gave us the global maximizer. An advantage of the geometrical technique is that it can also verify a global maximum, when one exists, for non-concave functions. Consider example 1 discussed in the previous section, where a price-taking two-product firm has a cubic profit function given by

$$(22) \qquad \pi(q_1,q_2) = (p_1-\alpha_1)q_1 + (p_2-\alpha_2)q_2 - F + \frac{1}{2}\beta_1 q_1^2 + \frac{1}{2}\beta_2 q_2^2 - \frac{1}{3}\gamma_1^3 - \frac{1}{3}\gamma_2^3 .$$

[10] One can also illustrate comparative static effects with the geometrical approach. For example, an increase in p_1 will shift the $\pi_1 = 0$ locus in figure 1 upward causing the value of $\bar{q}_1$ to rise and the value of $\bar{q}_2$ to fall. An increase in p_2 will shift the $\pi_2 = 0$ locus in figure 1 rightward causing the value of $\bar{q}_2$ to rise and the value of $\bar{q}_1$ to fall.

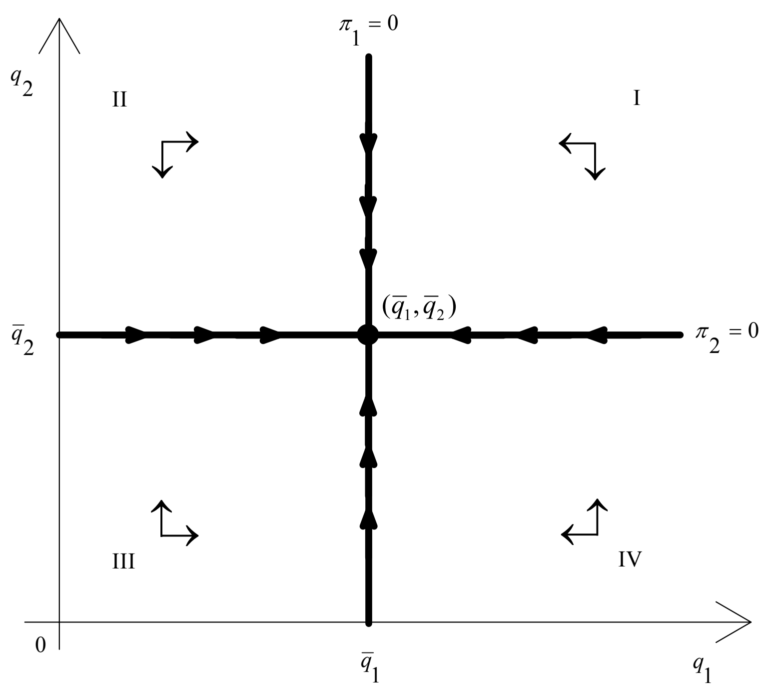
As shown in the previous section the profit function given by equation (22) is neither concave nor quasiconcave.

The first order conditions for maximizing profit in this case are:

$$\pi_i(q_1, q_2) = p_i - \alpha_i + \beta_i q_i - \gamma_i q_i^2 = 0;\ i = 1,2. \tag{23}$$

Assuming that $p_i > \alpha_i$ for $i = 1,2$, the single solution to the first order conditions is $\bar{q}_i = \dfrac{\beta_i + \sqrt{\beta_i^2 + 4\gamma_i(p_i - \alpha_i)}}{2\gamma_i}$ for $i = 1,2$. The graphs of the two first order conditions for this example are shown in figure 2. The $\pi_1 = 0$ locus is a vertical line at $\bar{q}_1$ and the $\pi_2 = 0$ locus is a horizontal line at $\bar{q}_2$.

Figure 2



It is easy to verify from figure 2 that the critical point $(\bar{q}_1, \bar{q}_2)$ is a global profit maximizer. From equation (22) note that $\pi_1 > 0$ for $0 \le q_1 < \bar{q}_1$ and $\pi_1 < 0$ for $q_1 > \bar{q}_1$. Thus, to the left of the $\pi_1 = 0$ locus all the horizontal arrows point in a rightward direction and to the right of the $\pi_1 = 0$ locus all the horizontal arrows point in a leftward direction. Also note that $\pi_2 > 0$ for $0 \le q_2 < \bar{q}_2$ and $\pi_2 < 0$ for $q_2 > \bar{q}_2$. Thus, below the $\pi_2 = 0$ locus all the vertical arrows point in an upward direction and above the $\pi_2 = 0$ locus all the vertical arrows point in a downward direction.

In region I of figure 2, the gradients all point in a southwest direction, in region II the gradients all point in a southeast direction, in region III the gradients all point in a northeast direction and in region IV the gradients all point in a northwest direction. At any point on the $\pi_1 = 0$ locus other than the critical point, the gradients point vertically towards the critical point and at any point on the $\pi_2 = 0$ locus other than the critical point, the gradients point horizontally towards the critical point.

It is important to note in figure 2 that if a gradient path touches the vertical $\pi_1 = 0$ locus or the horizontal $\pi_2 = 0$ locus before it reaches the critical point, the gradient path will remain on the $\pi_1 = 0$ or $\pi_2 = 0$ locus until the path reaches the critical point. Therefore, if one starts at any point in region I of figure 2, the gradient path must lead to $(\overline{q}_1, \overline{q}_2)$ without passing into regions II or III. Similarly if one starts at any point in regions II, III or IV, the gradient path must lead to $(\overline{q}_1, \overline{q}_2)$ without passing into another region. Because all points in the domain other than $(\overline{q}_1, \overline{q}_2)$ have gradient paths ending at $(\overline{q}_1, \overline{q}_2)$, $(\overline{q}_1, \overline{q}_2)$ is the global maximum.

An alternative way of finding the global maximum for this problem would be to recognize that because $\pi_1 < 0$ for $q_1 > \overline{q}_1$ and $\pi_2 < 0$ for $q_2 > \overline{q}_2$, one can make the domain closed and bounded by the restrictions that $0 \le q_1 \le \overline{q}_1$ and $0 \le q_2 \le \overline{q}_2$. Then apply the extreme value theorem methods to show that $(\overline{q}_1, \overline{q}_2)$ is the global maximizer. However, as the next example demonstrates, this method won't work in cases where there are no finite values for q_1 and q_2, say $\hat{q}_1$ and $\hat{q}_1$, where $\pi_1 < 0$ for $q_1 > \hat{q}_1$ and $\pi_2 < 0$ for $q_2 > \hat{q}_2$.

For the final example of the geometrical method, consider a special case of example 3 in discussed in the previous section. A two-product monopoly with a cost function $C(q_1, q_2) = F + a_1 q_1 + a_2 q_2$ faces inverse demand functions given by:

$$(24) \qquad p_1 = \alpha_1 - \beta_1 q_1^2 + .5q_2$$

$$(25) \qquad p_2 = \alpha_2 + .5q_1 - \beta_2 q_2^2 .$$

From the inverse demand functions above note that the two products produced by the monopoly are complements.

The profit function in this example is

$$(26) \qquad \pi(q_1, q_2) = (\alpha_1 - a_1)q_1 + (\alpha_2 - a_2)q_2 + q_1 q_2 - \beta_1 q_1^3 - \beta_2 q_2^3 .$$

The first order conditions for profit maximization are

$$(27) \qquad \pi_1 = \alpha_1 - a_1 + q_2 - 3\beta_1 q_1^2 = 0$$

$$(28) \qquad \pi_2 = \alpha_2 - a_2 + q_1 - 3\beta_2 q_2^2 = 0 .$$

It is easy to verify from the second derivatives that the profit function in this example is not concave. At $q_1 = q_2 = 0$, $\pi_{11} = \pi_{22} = 0$ and $\pi_{12} = 1$, hence the necessary condition for concavity $\pi_{11}\pi_{22} \ge \pi_{12}^2$ is violated.[11] Nevertheless, it is easy to use the geometrical method to verify that the intersection of the two first order conditions gives the global profit maximzer.

Figure 3

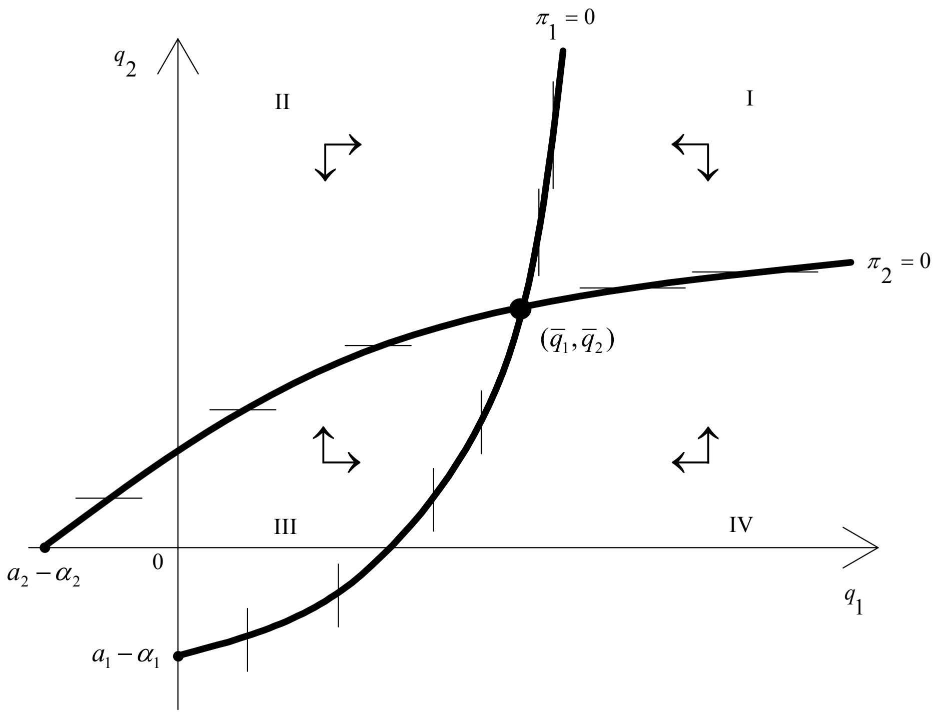


Figure 3 shows the graphs of these two first order conditions in the (q_1, q_2) plane. It is assumed in figure 3 that $\alpha_1 > a_1$ and $\alpha_2 > a_2$.The single intersection of the first order conditions is the unique critical point of the profit function and is labeled $(\bar{q}_1, \bar{q}_2)$ in figure 3. The intersecting first order conditions in figure 3 divide the domain into 4 regions labeled I through IV. Starting at any point in regions I or III (including the borders), the gradient paths will lead to the critical point $(\bar{q}_1, \bar{q}_2)$ without passing into either region II or IV. Starting at any point in either region II or IV, the gradient paths will either directly approach the critical point $(\bar{q}_1, \bar{q}_2)$ or the gradient paths will move into either region I or III where they will remain until the reach the critical point. Because all points in the domain other than the critical point have gradients paths which end at the critical point, the critical point $(\bar{q}_1, \bar{q}_2)$ in figure 3 is the global maximizer.

Conclusion

This paper analyzes and discusses some of the serious difficulties that arise in verifying the global maximum for a *seemingly* simple economic problem, profit maximization for a two-product firm. The most common procedure for justifying a global profit maximum in economics is to either establish or assume that the profit function is differentiable and concave, so that the satisfaction of the first order conditions guarantees a global profit maximum. This paper demonstrates, however, that many plausible profit functions are not concave nor even quasiconcave. A new gradient path geometrical method of verifying a

[11] Trying to prove that this function is quasiconcave by showing that $-\pi_1^2\pi_{22} + 2\pi_1\pi_2\pi_{12} - \pi_2^2\pi_{11} \geq 0$ at all points in the domain appears to be an intractable problem.

global maximum is presented that is analogous to the use of phase diagrams to solve for the equilibrium of a first-order differential equation system in two variables. An important advantage of the geometrical approach to optimization problems is that it always verifies a global maximum when the objective function is concave and it also verifies a global maximum in many cases where the objective function is neither concave nor quasiconcave.

References

Alpha C. Chiang and Kevin Wainwright, Fundamental Methods of Mathematical Economics, fourth edition, New York, 2005.

Raymond C. Battalio and Robert B. Ekelund,Jr., “Output Change under Third Degree Discrimination,” The Southern Economic Journal, 1972, 39, 285-290.

John P. Formby, Stephen K Layson, and W. James Smith, “The Law of Demand, Positive Sloping Marginal Revenue and Multiple Profit Equilibria,” Economic Inquiry, April 1982, 303-311.

Stephen K. Layson, “Verifying A Global Maximum,” Working Papers in Economics #ECO011101, Center for Applied Research, UNCG, 2002.

Olvi L. Mangasarian, “Pseudo-Convex Functions,” Journal of SIAM Control, Ser. A, 1965, 3, 281-291.

B. Martos, “The Direct Power of Adjacent Vertex Programming Methods,” Management Science, 1965, 12, 241-252.

J. Ponstein, “Seven Kinds of Convexity,” SIAM Review, 1967, 9, 115-19.
Paul A. Samuelson, Foundations of Economic Analysis, Atheneum, New York, 1972.

Eugene Silberberg and Wing Suen, The Structure of Economics: A Mathematical Analysis, Irwin McGraw-Hill, New York, 2001.

Carl P. Simon and Lawrence Blume, Mathematics for Economists, Norton, New York 1994.

Knut Sydsaeter and Peter Hammond, Mathematics for Economic Analysis, Prentice Hall, Englewood Cliffs, 1995.

Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strom, Further Mathematics for Economic Analysis, Prentice Hall,Edinburgh Gate, 2005.