JOURNAL OF ECONOMICS AND FINANCE EDUCATION • Volume 4 • Summer 2005 • Number 1

Duration and Bond Price Volatility: Some Further Results

Hassan Shirvani[1] and Barry Wilbratte[2]

Abstract

This paper evaluates the relative performances of three alternative bond price volatility approximations: Macaulay duration, convexity-augmented Macaulay duration, and an alternative proposed by the authors. Unlike the first two methods, the third is exact rather than approximate in predicting changes in zero-coupon bond prices. In addition, the alternative improves on the accuracy of the Macaulay formula and, for large interest rate changes, of the convexity-augmented formula. For small changes, the accuracy of our proposed and of convexity are similar. Finally, calculation of price changes is considerably more computationally efficient using our proposed formulation, providing it certain pedagogical advantages over convexity. (JEL: G12)

Introduction

In a recent paper (Shirvani and Wilbratte, 2002), we proposed an approximation formula for bond price volatility that represents an alternative to the well-known Macaulay (1938) formula. Like its Macaulay counterpart, this alternative relies only on the concept of bond duration (D). However, our formula provides more accurate approximations to the magnitude of coupon bond price changes in response to changes in interest rates. In addition, it resolves a deficiency in the Macaulay formula by providing a functional form which reflects the asymmetric effects of interest rate changes on bond prices.

Our earlier paper also compared the performance of our formula with the convexity-augmented Macaulay formula. Using several numerical examples, we showed that while the convexity-augmented Macaulay formula yields better estimates of coupon bond price volatility than ours for selected rate changes, the gains in accuracy are generally not great enough to warrant the tedious task of calculating bond convexities. While the convexity approach involves extended calculations, our alternative can be computed in seconds using a hand-held calculator. Students thus derive value from this method because its computational efficiency renders it a tool for use in the classroom. In addition, our formulation provides exact results when applied to zero coupon bonds, whereas the convexity approach provides only an approximation.

The purpose of this note is to advance our earlier discussion by demonstrating that, contrary to our previous perception, our proposed method yields results for coupon bonds which are in some cases roughly as accurate as the convexity-augmented formula and in many other cases substantially more accurate. Using numerical examples and graphic analysis to provide a more complete range of values than the numerical examples, this note shows that for large increases in interest rates, our formula actually outperforms the convexity-based Macaulay formula, and for moderate increases, the two are roughly equal in accuracy.

The following section demonstrates the foregoing points by describing and evaluating the relative performance of the three alternative measures of bond price volatility. The final section concludes.

Alternative Bond Price Volatility Approximation Formulas

The comparative performance of alternative formulas for approximating the extent of bond price volatility can best be assessed in terms of the relationship between the percentage change in the bond price

[1] Professor of Economics and Finance, University of St. Thomas, Houston, TX 77006, email shirvani@stthom.edu.
[2] Professor of Economics, University of St. Thomas, Houston, TX 77006, email wilbratt@stthom.edu.

and the percentage change in the bond yield (defined as one plus the yield to maturity of the bond). Differentiating the standard bond price formula with respect to the bond yield, we derive the following expression for the percentage change in the bond price:

(1) $$dP/P = - (\Sigma tCF_t/y^t/\Sigma CF_t/y^t)\ dy/y,$$

where P = the bond price, CF_t = the cash flow in period t, and y = the bond yield. Given that the expression in the parentheses on the right hand side is the bond duration, the above relationship is reduced to the well-known Macaulay formula:

(2) $$dP/P = - D\ (dy/y),$$

which is an exact relationship for infinitesimal changes in the bond yield. For finite changes in the bond yield, however, the value of D varies, rendering the above relationship only a rough approximation even if the underlying assumption of a horizontal yield curve holds.

The true relationship between finite changes in the coupon bond price and yield is depicted in Figure1, in which it is compared to the relationships derived from the three approximations discussed in this paper. For purposes of this comparison, we consider a 25-year bond with returns which are reflective of market conditions of early 2005, i.e., with a 5.25 percent coupon and a yield to maturity of 5 percent.

The baseline values plotted in Figure 1 are those of the true relationship, depicted by the dashed line. To derive the true relationship, we simply compute the market value of the bond using the 5 percent yield to maturity as a discount rate and then repeat the computations using higher and lower interest rates. The approximations are as follows: “Macaulay,” the approximation described above; “Convexity,” the convexity-augmented approach, presented below as equation 5; and “SW” (Shirvani-Wilbratte), the approximation proposed in this paper, given by equation 6, below.

As indicated by Figure 1, the true relationship is represented by a complex hyperbolic form, which is asymmetric, becoming steeper in the northwest quadrant and flatter in the southeast quadrant. In contrast, the Macaulay approximation is linear and symmetric. Consequently, the Macaulay approximation becomes increasingly inaccurate as the percentage change in the bond yield becomes larger in absolute value, and especially so for positive changes in the yield.

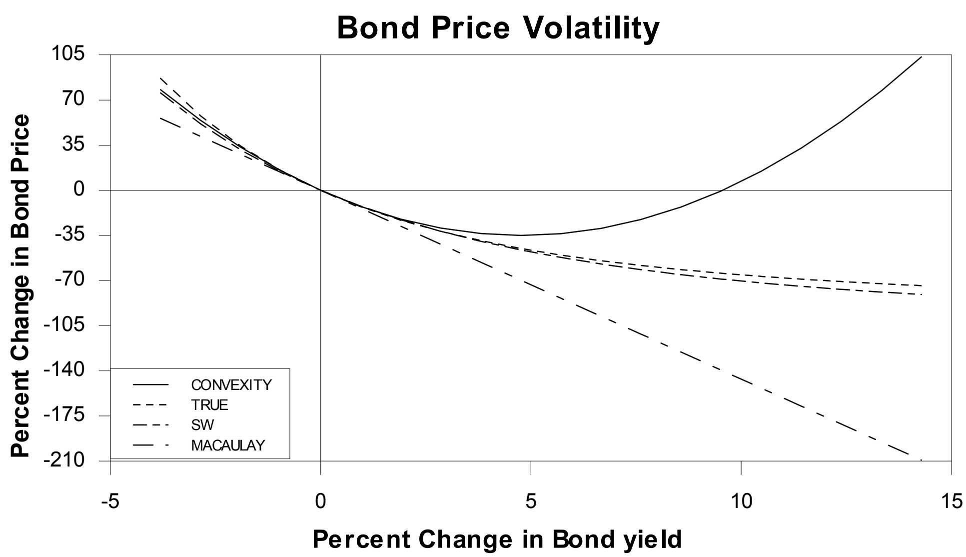

Figure 1

For purposes of illustrating the effects of finite changes in bond yields, we restate the Macaulay formula for bond price changes as follows:

(3) $$\Delta P/P \cong - D\ (\Delta y/y),$$

where Δ represents a finite change and $\cong$ indicates approximately equal.

Because of the inaccuracies of the Macaulay formulation, it is common in advanced treatments to augment the right-hand-side of the above equation by a corrective term to gain accuracy. The accepted correction to the Macaulay approximation is achieved by augmenting the Macaulay formula with the concept of convexity (C), defined as the second derivative of the bond price with respect to the bond yield. The equation for this approximation formula, based on the first two terms of the Taylor series expansion of the bond price equation, can be written as:

(4) $$\Delta P/P \cong - D\,(\Delta y/y) + \tfrac{1}{2}\, C\,(\Delta y/y)^2,$$

where

(5) $$C = [\Sigma t(t + 1)CF_t/y^t]/\ \Sigma CF_t/y^t$$

is the expression for bond convexity. (Most investment textbooks use the modified versions of D and C by dividing the versions used in this paper by y and y^2, respectively. However, we employ the unmodified versions as they relate ΔP/P to Δy/y and thus facilitate the comparisons of the above approximation formulas with our own, as demonstrated below.)

As Figure 1 indicates, the convexity-augmented Macaulay formula takes the form of a parabola passing through the origin. While this parabola improves on Macaulay estimates of bond price changes for limited variations in yields, it suffers from the shortcoming that for large increases in yields, it indicates positive changes in the bond price, a result obviously at odds with the inverse relationship between the two. Of greater empirical importance, the convexity approximation deteriorates as changes in market interest rates become significant, i. e., before the range where the slope of the parabola becomes positive. The variations in the accuracy of this approach are quantified later in this paper.

The alternative we propose approximates the change in bond prices using the following hyperbolic expression:

(6) $$\Delta P/P \cong (1 + \Delta y/y)^{-D} - 1$$

The derivation of the above expression is justified on the ground that any coupon bond has a "sister" zero-coupon bond with the same duration and price (Shirvani and Wilbratte, 2002). This result builds on the fact that duration is the center of gravity of a bond. The intuition of this fact can be appreciated with reference to an analogous application of the center of gravity concept in physics. To measure the gravitational pull between two massive and extended bodies, such as the earth and the moon, one must theoretically calculate the force of such a pull between any pair of particles belonging to these bodies, a next to impossible task. A simpler way to measure the pull is to use the fact that each of these massive bodies acts as though its entire mass is concentrated at its center of gravity, which for the earth and the moon, for example, will simply be at their centers. By thus treating each massively extended body as just a point of mass at its center of gravity, it becomes an easy task to calculate any gravitational pull for extended bodies. More generally, it can be shown that many properties of physical bodies, such as their stability and motion, can be more conveniently studied using the same center of gravity approach.

In a similar vein, as we have shown in our earlier paper, any bond can be likened to a flagpole with its center of gravity located at its duration. More specifically, any bond behaves as though its entire stream of cash payments over time were concentrated at a single point, its center of gravity, which is represented by the bond's duration. Thus, any coupon bond can be treated as a zero-coupon bond, with the maturity and hence duration of the zero-coupon bond being equal to the coupon bond's duration. In addition, the present values of all the payments on the two bonds, the coupon bond and its equivalent zero-coupon bond, will be the same, meaning the two bonds will have the same current price.

In short, the use of our center of gravity analogy will indicate that any coupon bond with duration D is equivalent to a zero-coupon bond with maturity D and with the same price. Under these conditions, for small changes in bond yields, where the size of the duration is not significantly impacted, the percentage change in the price of a coupon bond is roughly equal to the percentage change in the price of its sister zero-coupon bond, that is:

(7) $$\begin{aligned}\Delta P/P &\cong [M/(y + \Delta y)^D - M/y^D] / [M/y^D] \\ &= [1 + (\Delta y/y)]^{-D} - 1\end{aligned}$$

where M = the par value of the zero-coupon bond. Note that if the above equation is approximated by the first two terms of its Taylor expansion, we obtain the Macaulay formula as an approximation to our model. In addition, the third term of the Taylor expansion of our equation provides a rough approximation for the convexity of a coupon bond with duration D as equal to D(D + 1), which is the convexity of the sister zero-coupon bond with maturity D.

Students can be motivated to appreciate the importance of equation 6, as opposed to its more commonly used Macaulay rival, by first computing the percentage changes in the price of a given bond for equal but opposite and finite changes in yields. These calculations should clearly show that the bond price changes are not symmetric, rising more in response to decreases in yields than falling when yields increase. Next, using the Macaulay equation to approximate the same price changes, students are shown that the Macaulay approach will produce results which are, incorrectly, symmetric as well as rather inaccurate, especially for large yield changes. Finally, the use of equation 6, which relies only on the already computed duration value, will be shown to produce both asymmetric and better estimates of the actual price changes. For many undergraduate students, this will very much end the discussion. For graduate students, however, a discussion of the convexity concept may well be in order (see, for example, Reilly and Brown, 2003). Even here, students may be shown that the additional gains in accuracy obtained by the introduction of convexity may not justify the tedious extra calculations required.

As a further pedagogical motivation, students should be reminded that many investors hold bonds for the safety they provide, and thus astute investors will be interested in the degree of risk of capital loss which characterizes their bond investments. Equation 6 provides a measure of such risk by indicating the extent to which various bonds' prices change for a given change in bond yields. In addition, students can be shown that active bond traders can use equation 6 to determine the percentage gains or losses they will realize if bond yields fall or rise. Indeed, given the numerical value of a bond's duration, D, students can use hand held calculators to perform exercises in class using equation 6, as the computations require only a few strokes and can be completed in a matter of seconds.

Finally, returning to Figure 1, we see that the hyperbolic curve associated with our expression (SW) has the advantage that it continuously slopes downward, thus preserving the inverse relationship between the changes in coupon bond prices and their yields throughout the range of possible interest rates. In particular, for large interest rate changes, it provides a better approximation than the convexity-augmented Macaulay curve mentioned above. For smaller changes in interest rates, however, the convexity-augmented Macaulay curve performs slightly better. More generally, as Figure 1 demonstrates, in response to rising interest rates, the convexity formula initially outperforms (see "CONVEXITY"), then matches, and finally is outperformed by our formula (see "SW"). The interesting issues are, at what levels and to what extent do the alternatives outperform each other?

Table 1
Predicted Percentage Changes in Coupon Bond Prices

Interest Rate Change (Basis Points)	True Price Change (%Δ)	Macaulay Formula (%Δ)	Convexity-Augmented (%Δ)	Shirvani-Wilbratte (%Δ)
+500	-45.05	-69.78 (-24.72)	-35.01 (+10.04)	-49.42 (-4.37)
+300	-31.76	-41.87 (-10.11)	-29.35 (+2.41)	-33.82 (-2.06)
+100	-12.66	-13.96 (-1.29)	-12.56 (+0.10)	-12.97 (-0.30)
-100	+15.46	+13.96 (-1.50)	+15.35 (-0.11)	+15.05 (-0.41)
-300	+57.89	+41.87 (-16.02)	+54.38 (-3.51)	+52.92 (-4.97)
-500	+123.38	+69.78 (-53.60)	+104.54 (-18.84)	+104.40 (-18.98)

NOTE: Approximation Errors in Parentheses

To address these issues, we present a simple tabular example and compare the accuracy of the alternative formulations. Table 1 depicts the true percentage changes in the coupon bond price represented in Figure 1 for various changes in interest rates and compares them with the percentage changes indicated by the Macaulay model, the convexity-augmented Macaulay model, and ours (Shirvani-Wilbratte). As indicated in the table, the Macaulay model is increasingly inaccurate as compared with the alternatives for larger and larger positive changes in y. More specifically, the performances of the convexity model and our alternative are roughly equal for interest rate increases up to 3 percentage points (300 basis points). For positive interest rate changes in excess of 3 percentage points, our model is significantly more accurate. For negative changes, corresponding to Figure 1, the models are roughly equal in accuracy throughout.

Finally, turning from coupon to zero-coupon bonds, we demonstrate that our model is exact for the latter but that the Macaulay and convexity augmented formulations provide only approximations. The comparison appears in Table 2, using the same bond maturity as in Figure and Table 1, 25 years. As indicated in the second column, headed "True Price Change" and in the rightmost column, our approach provides the exact percentage change in the bond price. In contrast, the Macaulay formula becomes extremely inaccurate for 300 basis point increases, and the convexity-augmented formula improves on the Macaulay formula but becomes increasingly inaccurate for large interest rate changes, especially negative changes.

Table 2
Predicted Percentage Changes in Zero-Coupon Bond Prices

Interest Rate Change (Basis Points)	True Price Change (%Δ)	Macaulay Formula (%Δ)	Convexity-Augmented (%Δ)	Shirvani-Wilbratte (%Δ)
+500	-68.75	-119.05 (-50.30)	-45.35 (+23.40)	-68.75 (0)
+300	-50.55	-71.43 (-20.88)	-44.90 (+5.65)	-50.55 (0)
+100	-21.10	-23.81 (-2.71)	-20.86 (+0.24)	-21.10 (0)
-100	+27.03	+23.81 (-3.22)	+26.76 (-0.27)	+27.03 (0)
-300	+106.41	+71.43 (-34.98)	+97.96 (-8.45)	+106.41 (0)
-500	+238.64	+119.05 (-119.59)	+192.74 (-45.90)	+238.64 (0)

NOTE: Approximation Errors in Parentheses

Conclusion

This paper offers an expression for computing bond price volatility with advantages over the two accepted approaches. As compared with the Macaulay approach, ours is uniformly more accurate in estimating the extent of bond price changes with respect to changes in interest rates. Our approach is also more accurate than the convexity approach for large increases in interest rates but slightly less so for small changes. Furthermore, in the ranges in which the accuracy of our model and of the convexity approach is similar, the advantage of our model is simply its computational efficiency. This is useful for classroom demonstrations and can be applied by students even while taking examinations. In times when unusually low interest rates prevail or financial markets face a high degree of uncertainty due to events such as an increase in the size of the budget deficit or exchange rate volatility, it is arguable that the accompanying possibility of large interest rate changes enhances the value of our model as a measure of risk. Finally, unlike the Macaulay or the convexity approach, ours is exact for zero-coupon bonds.

References

Macaulay, F. R. 1983. *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the U.S. since 1856*. New York: National Bureau of Economic Research.

Reilly, F. K. and K.C. Brown. 2003. *Investment Analysis and Portfolio Management*, 7th edition. USA: South-Western Publishing.

Shirvani, Hassan, and Barry Wilbratte. 2002. "Two Pedagogical Simplifications of the Concept of Duration." *Journal of Economics and Finance Education* 1: 18-23.