Delta Gamma Hedging and the Black-Scholes Partial Differential Equation (PDE)

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Abstract

The objective of this paper is to examine the notion of delta-gamma hedging using simple stylized examples. Even though the delta-gamma hedging concept is among the most challenging concepts in derivatives, standard textbook exposition of delta-gamma hedging usually does not proceed beyond a perfunctory mathematical presentation. Issues such as contrasting call delta hedging with put delta hedging, gamma properties of call versus put delta hedges, etc., are usually not treated in sufficient detail. This paper examines these issues and then places them within the context of a fundamental result in derivatives theory - the Black-Scholes partial differential equation. Many of these concepts are presented using Excel and a simple diagrammatic framework that reinforces the underlying mathematical intuition.

Introduction

The notion of delta hedging is a fundamental idea in derivatives portfolio management. The simplest notion of delta hedging refers to a strategy whereby the risk of a long or short stock position is offset by taking an offsetting option position in the underlying stock. The nature and extent of the option position is dictated by the underlying sensitivity of the option's value to a movement in the underlying stock price (i.e. option delta). Since the delta of an option is a local first order measure, delta hedging protects portfolios only against small movements in the underlying stock price. For larger movements in the underlying price, effective risk management requires the use of both first order and second order hedging or delta-gamma hedging. In some cases, a third order approximation (delta-gamma-speed hedging) may also be required.

The objective of this paper is to examine the notion of delta-gamma hedging using simple stylized examples and to illustrate these concepts using Excel. Even though the delta-gamma hedging concept is among the most challenging concepts in derivatives portfolio management, standard textbook exposition of delta-gamma hedging usually does not proceed beyond a perfunctory mathematical presentation of delta hedging with calls. See Chance and Brooks (2010), Hull (2008), Kolb and Overdahl (2007), Chance (2003), Jarrow and Turnbull (2000). Issues such as delta hedging with puts, contrasting delta hedging with calls versus delta hedging with puts, gamma properties of call versus put delta hedges, etc. are usually not treated. This paper examines these issues and then places them within the context of a fundamental result in derivatives theory - the Black-Scholes partial differential equation (PDE). Many of these concepts are presented using a simple diagrammatic framework that highlights and reinforces the underlying conceptual and mathematical intuition.

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Option Greeks

Delta hedging is based on the notion of insulating portfolios from small movements in the underlying asset price by taking an offsetting option position. The option position is dictated by the sensitivity of the option value to underlying movements in the asset price. As the underlying stock price changes, the option's delta changes and the hedge must be re-calibrated to maintain its effectiveness. (See Hull (2008, p.363-366) for a detailed example of dynamically hedging against a short call position). The option's sensitivity to a change in the underlying variables such as the stock price, volatility, time to option maturity and the risk free rate is therefore crucial in hedging against different types of risk. These option sensitivities or option greeks can be derived from a standard, non-dividend paying, European type, Black-Scholes model of the form:

$$C = S\,N(d_1) - X\,e^{-r_f t}\,N(d_2) \qquad (1)$$

where C and S are the current values of the call and stock, $N(d_1)$ and $N(d_2)$ are cumulative unit normal probability distribution values, X is the exercise price, r_f is the risk free interest rate and t is the time to option maturity. The explicit form of $N(d_1)$ and $N(d_2)$ are given by:

$$d_1 = \frac{\ln (S/X) + r_f\ t}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\ ; \quad d_2 = d_1 - \sigma\sqrt{t}$$

where σ is the standard deviation of the continuously compounded asset return. The current value of a put (P) can be determined by applying put-call parity. Thus:

$$P = S\,[N(d_1) - 1] - X\,e^{-r_f t}\,[N(d_2) - 1] \qquad (2)$$

Now (1) implies that:

$$C = C\,(S, \sigma, r_f, t) \qquad (3)$$

Using a third order Taylor series expansion, (3) can be written as:

$$dC = \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial r_f}dr_f + \frac{1}{2}\frac{\partial^2 C}{(\partial S)^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 C}{(\partial \sigma)^2}(d\sigma)^2 + \frac{1}{2}\frac{\partial^2 C}{(\partial t)^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 C}{(\partial r_f)^2}(dr_f)^2 + \frac{\partial^2 C}{(\partial S)(\partial t)}(dS)(dt) + \frac{1}{6}\frac{\partial^3 C}{(\partial S)^3}(dS)^3 + \frac{1}{6}\frac{\partial^3 C}{(\partial S)^2(\partial \sigma)}(dS)^2(d\sigma) + \frac{1}{6}\frac{\partial^3 C}{(\partial S)^2(\partial t)}(dS)^2(dt) \qquad (4)$$

These derivatives or option greeks are easily derived from (4) as follows: Delta = $(\partial C/\partial S) = \Delta$; Vega = $(\partial C/\partial \sigma) = \nu$; Theta = $(\partial C/\partial t) = \theta$; Rho = $(\partial C/\partial r_f) = \rho$; Gamma = $(\partial^2 C/(\partial S)^2) = \Gamma$. One can define equivalent terms for put options[2]. These option greeks are crucial in the construction of hedging strategies. Their use is analyzed in the subsequent sections.

[2] Other Less Common Option Greeks Are: Charm = (∂□C/∂S∂T) = (∂Δ/∂T); Speed = (∂□C/∂S³) = (∂Γ/∂S); Volga = (∂□C/∂S²∂σ) = (∂Γ/∂σ), Color = (∂□C/∂S²∂T) = (∂Γ/∂□).The Terms Charm And Speed Are Borrowed From Names Used In Quantum Physics For Sub-Atomic Particles. See Chapter 8 In Neftci (2004) For A Detailed Treatment Of The Option Greeks.

Delta Hedging

Consider the following stylized example:

Current Price of Option 1 (S)	=	$100
Exercise Price of Option 1 (X)	=	$100
Risk Free Return (r_f)	=	5% p.a.
Time to Maturity (t)	=	91 days or 91/365 = 24.93%
Volatility (σ)	=	20% p.a.

The resulting Black-Scholes call and put prices for Option 1 are $4.61 and $3.37, respectively[3]. These prices, as well as the standard option greeks, are shown for two options – Option 1 and Option 2. (See Tables 1a and 1b. The Excel commands used to generate the values in Table 1a are shown in Table 1b). Both Option 1 and 2 are on the same stock but differ in their exercise prices. In the succeeding analyses, Option 1 values are used. Option 2 values are used in the subsequent section on delta/gamma hedging.

Suppose now that a portfolio manager wanted to delta hedge 1000 shares of a long stock position on ABC stock using Option 1 calls. Assume that we are looking at the hedge immediately after it has been instituted. Thus, time, volatility and the risk free rate are constant. The delta of this stock/call portfolio (Δ_p) is then given by:

$$\Delta_p = \eta_s \Delta_s + \eta_c \Delta_c \qquad (5)$$

where η_s refers to the number of shares in the stock portfolio, Δ_s is the delta of the stock (which is 1 since the value of the stock varies one to one with the stock price), η_c is the number of calls to be determined and Δ_c is the call delta which is equal to .5694. Setting the delta of the portfolio in (5) equal to zero creates a portfolio that is hedged against first-order movements in the underlying stock price. The number of long/short calls to be traded to create a delta-neutral hedge for a 1000 share portfolio can be easily solved from (5) thus:

$$0 = (1000)(1) + (\eta_c)\,(.5694)$$
$$\eta_c = -1756$$

Thus, 1756 Option 1 calls need to be sold in order to hedge a 1000 share portfolio or equivalently a short call position of 1756 calls can be hedged using a long stock position of 1000 shares[4]. The performance of this delta-

[3] In Table 1, Theta Is Computed On A Per Annum Basis. Thus, Call Theta For Option 1 Per Day Is Given By: (-10.4852)/(365) = -.0287.

[4] Suppose The Stock Price Declines To $99 (See Table 2). At $99, The Stock Portfolio Has Lost (-$1)(1000 Shares) Or -$1000. The Short Call Portfolio Has Gained About $965 Since 1756 Calls Were Sold At $4.61 And Purchased Back At $4.06 (The Black-Scholes Call Value At A Stock Price Of $99). The Net Change In The Portfolio Is Thus -$35. The Call And Delta Neutral Portfolio Values In Table 2 Are Generated Using Excel's What-If Analysis And Data Table Function.

Table 1a: Black-Scholes & Option Greeks

	A	B	C
1			
2		OPTION 1	OPTION 2
3	CURRENT STOCK PRICE	$100.00	$100.00
4	EXERCISE PRICE	$100.00	$110.00
5	RISK-FREE RATE	5.00%	5.00%
6	TIME TO MATURITY (91 Days)	24.93%	24.93%
7	VOLATILITY	20.00%	20.00%
8			
9	D1	0.1748	-0.7796
10	D2	0.0749	-0.8795
11			
12	N(D1)	0.5694	0.2178
13	N(D2)	0.5299	0.1896
14			
15	CALL PRICE	$4.61	$1.19
16			
17	PUT PRICE	$3.37	$9.82
18			
19			
20	CALL DELTA	0.5694	0.2178
21	CALL GAMMA	0.0393	0.0295
22	CALL VEGA	19.6179	14.6991
23	CALL THETA	-10.4852	-6.9255
24	CALL RHO	13.0464	5.1343
25			
26			
27			
28	PUT DELTA	-0.4306	-0.7822
29	PUT GAMMA	0.0393	0.0295
30	PUT VEGA	19.6179	14.6991
31	PUT THETA	-5.5471	-1.4936
32	PUT RHO	11.5763	21.9507
33			
34	CALL & PUT SPEED	0.0001	-0.0002
35			
36			
37	BLACK-SCHOLES PDE (CALL OPTION)	0.0000	0.0000
38	BLACK-SCHOLES PDE (PUT OPTION)	0.0000	0.0000
39			
40	NOTE: Theta is an a per annum basis. Thus, call theta for Option 1 per day is given by: (-10.4852)/(365) = -.0287.		
41			

Table 1b: Excel Commands used to Generate Table 1a

	A	B	C
1			
2		OPTION 1	OPTION 2
3	CURRENT STOCK PRICE	100	100
4	EXERCISE PRICE	100	110
5	RISK-FREE RATE	0.05	0.05
6	TIME TO MATURITY (91 Days)	=91/365	=91/365
7	VOLATILITY	0.2	0.2
8			
9	D1	=((LN(B3/B4) + B5*B6) / (B7*SQRT(B6))) + (0.5*B7*SQRT(B6))	=((LN(C3/C4) + C5*C6) / (C7*SQRT(C6))) + (0.5*C7*SQRT(C6))
10	D2	=B9-(B7*SQRT(B6))	=C9-(C7*SQRT(C6))
11			
12	N(D1)	=NORMSDIST(B9)	=NORMSDIST(C9)
13	N(D2)	=NORMSDIST(B10)	=NORMSDIST(C10)
14			
15	CALL PRICE	=B3*B12-B4*EXP(-B5*B6)*B13	=C3*C12-C4*EXP(-C5*C6)*C13
16			
17	PUT PRICE	=B3*(B12-1) + B4*(EXP(-B5*B6))*(1-B13)	=C3*(C12-1) + C4*(EXP(-C5*C6))*(1-C13)
18			
19			
20	CALL DELTA	=B12	=C12
21	CALL GAMMA	= (1/SQRT(2*PI())) * EXP(-0.5*B9*B9) * 1/(B3*B7*SQRT(B6))	= (1/SQRT(2*PI())) * EXP(-0.5*C9*C9) * 1/(C3*C7*SQRT(C6))
22	CALL VEGA	= B3 * SQRT(B6) * (1/SQRT(2*PI())) * EXP(-0.5*B9*B9)	= C3 * SQRT(C6) * (1/SQRT(2*PI())) * EXP(-0.5*C9*C9)
23	CALL THETA	=-((B3*EXP(-0.5*B9*B9)*B7) / (2*SQRT(2*PI()*B6))) - (B5*B4*EXP(-B5*B6)*B13)	=-((C3*EXP(-0.5*C9*C9)*C7) / (2*SQRT(2*PI()*C6))) - (C5*C4*EXP(-C5*C6)*C13)
24	CALL RHO	=B4 * B6* EXP(-B5*B6) *B13	=C4 * C6* EXP(-C5*C6) *C13
25			
26			
27			
28	PUT DELTA	=B12-1	=C12-1
29	PUT GAMMA	= (1/SQRT(2*PI())) * EXP(-0.5*B9*B9) * 1/(B3*B7*SQRT(B6))	= (1/SQRT(2*PI())) * EXP(-0.5*C9*C9) * 1/(C3*C7*SQRT(C6))
30	PUT VEGA	= B3 * SQRT(B6) * (1/SQRT(2*PI())) * EXP(-0.5*B9*B9)	= C3 * SQRT(C6) * (1/SQRT(2*PI())) * EXP(-0.5*C9*C9)
31	PUT THETA	=-((B3*EXP(-0.5*B9*B9)*B7) / (2*SQRT(2*PI()*B6))) + (B5*B4*EXP(-B5*B6)*(1-B13))	=-((C3*EXP(-0.5*C9*C9)*C7) / (2*SQRT(2*PI()*C6))) + (C5*C4*EXP(-C5*C6)*(1-C13))
32	PUT RHO	=-B4 * B6* EXP(-B5*B6) *(B13-1)	=-C4 * C6* EXP(-C5*C6) *(C13-1)
33			
34	CALL & PUT SPEED	=((B9 + B7*SQRT(B6))/(B3))*B21	=((C9 + C7*SQRT(C6))/(C3))*C21
35			
36			
37	BLACK-SCHOLES PDE (CALL OPTION)	=$B$23-($B$5)*($B$15) +($B$5)*($B$3)*($B$20) + (0.5)*($B$21)*($B$7)*($B$7)*($B$3)*($B$3)	=$C$23-($C$5)*($C$15) +($C$5)*($C$3)*($C$20) + (0.5)*($C$21)*($C$7)*($C$7)*($C$3)*($C$3)
38	BLACK-SCHOLES PDE (PUT OPTION)	=$B$31-($B$5)*($B$17) +($B$5)*($B$3)*($B$28) + (0.5)*($B$29)*($B$7)*($B$7)*($B$3)*($B$3)	=$C$31-($C$5)*($C$17) +($C$5)*($C$3)*($C$28) + (0.5)*($C$29)*($C$7)*($C$7)*($C$3)*($C$3)
39			
40			

neutral hedge with calls is shown in Table 2 and graphed in Figure 1[5]. Notice that the hedge performs increasingly poorly the further the stock price moves away from the initial stock price of $100. This is not difficult to understand given that a long stock position is being hedged using a short call position. As the stock price declines, the stock position incurs higher and higher losses. At the limit at a stock price of $0, the stock position loses $100,000. The maximum gain on the short call position can however never exceed ($4.61)(1756 calls) or $8093. The asymmetric nature of the return on the short call position ensures that it performs poorly for large deviations away from the initial stock price.

It is also instructive to consider the portfolio gamma of the long stock/short call portfolio. The gamma of the call option is the second derivative of (1) with respect to the stock price. Thus:

$$\frac{\partial^2 C}{\partial^2 S} = \frac{\partial^2 P}{\partial^2 S} = \frac{1}{\sqrt{2\prod}} \quad [e^{(-d_1^2/2)}] \quad \frac{1}{S\sigma\sqrt{t}} \qquad (6)$$

The symmetry of the unit normal distribution ensures that call and put gammas are identical. The portfolio gamma [Γ_p] of the long stock/short call portfolio is then given by:

$$\Gamma_p = \eta_s \Gamma_s + \eta_c \Gamma_c \qquad (7)$$

where Γ_s is the gamma of the stock (equal to zero) and Γ_c is the gamma of the call (equal to .0393; see values for Option 1 in Table 1). The portfolio gamma is then equal to:

[5] We Assume An Instantaneous Change In Stock Prices From The Initial Value Of $100. This Enables One To Focus On The Effect Of Stock Price Changes Keeping Constant The Effect Of A Change In Other Variables Such As Volatility Or Option Maturity. For Instance, We Could Easily Analyze Hedge Performance After The Lapse Of A Week. The Delta Hedge Will, Of Course, Perform Worse Than The Reported Results Here Since Theta Risk Now Becomes A Factor.

$$\Gamma_p = (1000)(0) + (.0393)(-1756) = -69$$

The negative convexity of the long stock/short call portfolio provides the underlying rationale for the delta neutral portfolio function described by Figure 1. The negative convexity of this portfolio explains its poor hedging performance.

The next case to be considered is delta hedging with puts. Following a procedure similar to the above and noting that the put delta on Option 1 is -.4306, we can determine that the number of puts to be purchased to create a delta neutral hedge for 1000 shares is about 2322 puts. The long position in puts offsets the decrease in stock portfolio value as the stock price declines. The performance of the delta neutral hedge with puts is shown in Table 3 and Figure 1. It is immediately apparent that this hedge performs considerably better than the hedge with short calls. The intuitive reason is that as the stock price declines the long put position moves deeper and deeper into the money. At the limit, when the stock price is $0, the stock portfolio loses $100,000 whereas the put position gains $98.75 per put or about $229,298 for the entire put position[6]. The same notion is reinforced by examining the gamma of this portfolio which is given by:

$$\Gamma_p = (1000 \text{ shares})\ (0) + (2322 \text{ puts})\ (.0393) = 91$$

The positive gamma of this portfolio, made evident in the put portfolio function depicted in Figure 1, reinforces the superior performance of the delta neutral hedge with puts as compared to the delta neutral hedge with calls. For any price deviation from the initial price of $100, the delta hedge with puts would clearly be a superior choice.

Table 2: Delta Hedging with Calls

CURRENT STOCK PRICE	$100.00	**STOCK PRICE**	**CALL VALUE**	**VALUE OF THE DELTA NEUTRAL PORTFOLIO**
EXERCISE PRICE	$100.00	$100	$4.61	$0
RISK-FREE RATE	5.00%	$85	$0.27	-$7,375
TIME TO MATURITY (91 DAYS)	24.93%	$86	$0.35	-$6,519
VOLATILITY	20.00%	$87	$0.45	-$5,696
		$88	$0.57	-$4,911
		$89	$0.72	-$4,170
		$90	$0.89	-$3,477
		$91	$1.10	-$2,836
D1	0.1748	$92	$1.34	-$2,253
D2	0.0749	$93	$1.61	-$1,732
		$94	$1.92	-$1,275
N(D1) [CALL DELTA]	0.5694	$95	$2.27	-$886
N(D2)	0.5299	$96	$2.65	-$566
PUT DELTA	-0.4306	$97	$3.08	-$317
		$98	$3.55	-$140
CALL PRICE	$4.61	$99	$4.06	-$35
		$100	$4.61	$0
PUT PRICE	$3.369	$101	$5.20	-$34.21
		$102	$5.82	-$135
LONG SHARES IN THE PORTFOLIO	1000	$103	$6.49	-$300
SHORT CALLS REQUIRED FOR HEDGING	1756.3397	$104	$7.19	-$527
PORTFOLIO DELTA	0	$105	$7.92	-$810
INITIAL VALUE OF STOCK PORTFOLIO	$100,000	$106	$8.68	-$1,147
INITIAL VALUE OF SOLD CALLS	$8,093	$107	$9.47	-$1,534
		$108	$10.28	-$1,967
		$109	$11.12	-$2,441
		$110	$11.98	-$2,952
CALL Z (d^2)	0.0153	$111	$12.86	-$3,497
CALL / PUT GAMMA	0.03934	$112	$13.76	-$4,073
PORTFOLIO GAMMA	-69.10069	$113	$14.67	-$4,675
PORTFOLIO THETA	18,415.4941	$114	$15.60	-$5,300
		$115	$16.53	-$5,946

[6] At A Stock Price Of $0, The Value Of The Put At Option Maturity Equals Its Exercise Price Of $100. The Discounted Present Value Is Equal To [$100] [($E^{-(.05)(.2493)}$] = $98.76.

Figure 1

Performance of Call & Put Delta Neutral Portfolios

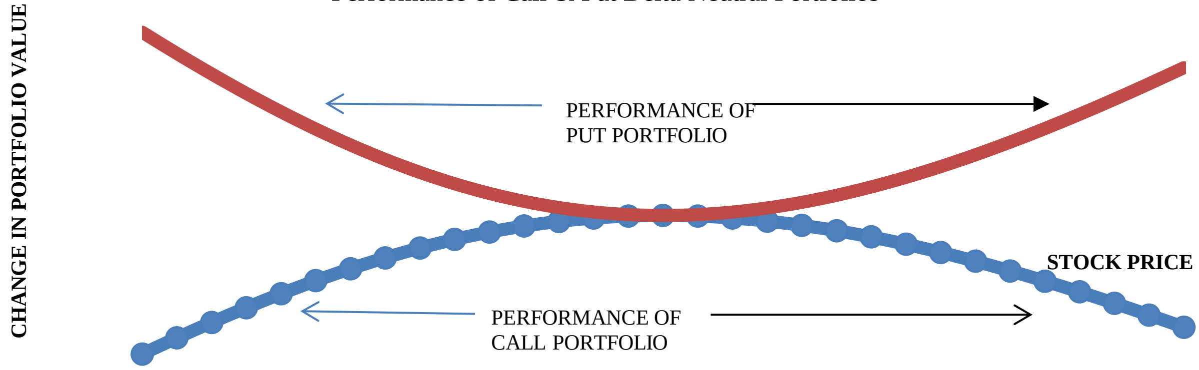

Table 3: Delta Hedging with Puts

CURRENT STOCK PRICE	$100.00
EXERCISE PRICE	$100.00
RISK-FREE RATE	5.00%
TIME TO MATURITY (91 DAYS)	24.93%
VOLATILITY	20.00%
D1	0.1748
D2	0.0749
N(D1) [CALL DELTA]	0.5694
N(D2)	0.5299
PUT DELTA	-0.4306
CALL PRICE	$4.61
PUT PRICE	$3.37
LONG SHARES IN THE PORTFOLIO	1000
LONG PUTS REQUIRED FOR HEDGING	2322.1572
PORTFOLIO DELTA	0
INITIAL VALUE OF STOCK PORTFOLIO	$100,000
INITIAL VALUE OF PURCHASED PUTS	$7,823
CALL Z (d^2)	0.0153
CALL / PUT GAMMA	0.0393
PORTFOLIO GAMMA	91.36
PORTFOLIO THETA	-12,881.2310
BLACK-SCHOLES PORTFOLIO PDE	0.0000

STOCK PRICE	PUT VALUE	VALUE OF THE DELTA NEUTRAL PORTFOLIO
$100	$3.37	$0
$85	$14.03	$9,751
$86	$13.11	$8,618
$87	$12.21	$7,530
$88	$11.33	$6,493
$89	$10.48	$5,513
$90	$9.65	$4,597
$91	$8.86	$3,750
$92	$8.10	$2,979
$93	$7.37	$2,290
$94	$6.68	$1,686
$95	$6.03	$1,171
$96	$5.41	$748
$97	$4.84	$419
$98	$4.31	$186
$99	$3.82	$46
$100	$3.37	$0
$101	$2.96	$45
$102	$2.58	$179
$103	$2.25	$397
$104	$1.95	$696
$105	$1.68	$1,071
$106	$1.44	$1,517
$107	$1.23	$2,028
$108	$1.04	$2,600
$109	$0.88	$3,227
$110	$0.74	$3,903
$111	$0.62	$4,624
$112	$0.52	$5,385
$113	$0.43	$6,181
$114	$0.36	$7,007
$115	$0.29	$7,861

Delta-Gamma Hedging

The portfolio considered above is clearly not hedged against all types of risk. In addition to large movements in the stock price it is susceptible to changes in volatility and the risk free rate. The time decay of the option introduces another source of risk. If the underlying asset exhibits wide price swings, both first order (delta) and second order (gamma) movements need to be taken into consideration. Suppose we now extend the previous example of a delta hedge with calls to delta-gamma hedging with calls. The parameters of the option we analyzed earlier [Option 1] were as follows: Call Delta = .5694; Call Gamma = .0393. To create a delta-gamma hedge, we need a second option on the same stock. Let the exercise price of this second option [Option 2] be $110 with all other parameters being the same as Option 1. The call delta and gamma of Option 2 can be determined to be .2178 and .0295, respectively (see Table 1). To make a 1000 share portfolio (η_s = 1000) delta-gamma neutral, the following set of simultaneous equations needs to be solved where the first equation imposes delta neutrality and the second equation imposes gamma neutrality.

$$\Delta_p = \eta_s \Delta_s + \eta_1 \Delta_1 + \eta_2 \Delta_2 = 0 \quad (8)$$

$$\Gamma_p = \eta_s \Gamma_s + \eta_1 \Gamma_1 + \eta_2 \Gamma_2 = 0 \quad (9)$$

The 1 and 2 subscripts refer to Option 1 and Option 2 values. Solving simultaneously yields η_1 = –3588 and η_2 = 4789. Thus, to ensure delta-gamma neutrality for this portfolio, sell 3588 of Option 1 calls and buy 4789 of Option 2 calls. The performance of this portfolio at different stock prices is reported in Table 4[7]. Using a similar procedure, it can be easily shown that delta-gamma neutrality using puts requires selling 1630 of Option 1 puts and buying 2176 of Option 2 puts.

The performance of both the call and put delta-gamma neutral portfolios are compared in Figure 2. Figure 2 provides good intuitive insight into the conceptual notion behind delta-gamma hedging. When comparing Figures 1 and 2, it is immediately evident that delta-gamma portfolios preserve values for much larger swings in the underlying stock price as compared to delta neutral portfolios. For instance, at a stock price of $90, the value of the delta neutral portfolio with calls decreases by $3477 but the delta-gamma neutral portfolio with calls decreases by about half as much or $1811. For relatively small deviations from the initial stock price of $100, both the delta-gamma neutral call and put portfolios perform in a fairly similar manner. However, for larger deviations in the underlying price the performance of the delta-gamma call and put portfolios differ significantly. The delta/gamma call portfolio has a significantly greater upside but also a far greater downside as compared to the delta/gamma put portfolio. The unlimited upside potential of calls combined with the limited downside protection offered by short call positions explains the delta-gamma call function. The delta-gamma put portfolio offers limited upside value but the protective nature of long put positions implies that while the delta-gamma put function has limited upside value it offers substantial downside protection as compared to the delta-gamma call function.

The Black-Scholes Partial Differential Equation (PDE) and Delta-Gamma Hedges

In this section, the discussion on first and second order hedges are placed within the context of a fundamental result in derivatives theory – the Black-Scholes PDE - and the analysis shows explicitly the manner in which this fundamental relationship is satisfied. The B-S PDE demonstrates the manner in which an asset that is delta hedged by buying and selling the underlying option in just the right proportions results in a risk free portfolio. The Black-Scholes PDE can be written as:

[7] The Change In Portfolio Value Is Calculated Thus. At A Stock Price Of $95 (See Table 4), The Loss On The Stock Portfolio Is ($95-$1000) (1000 Shares) = $5,000. The Profit/Loss On The Option 1 Trade Equals ($4.61-$2.27) (3588 Calls) = $8396. The Profit/Loss On The Option 2 Trade Equals ($.42-$1.19) (4789 Calls) = -$3688. Thus, The Approximate Net Change In Portfolio Value Equals -$292. The Exact Value Calculated In Table 4 Equals -$261.

$$\theta_p - r_f C_p + r_f S \Delta_p + \frac{1}{2}\Gamma_p \sigma^2 S^2 = 0 \qquad (10)$$

Figure 2
Performance of the Delta-Gamma Neutral Portfolios
(Call versus Put Portfolios)

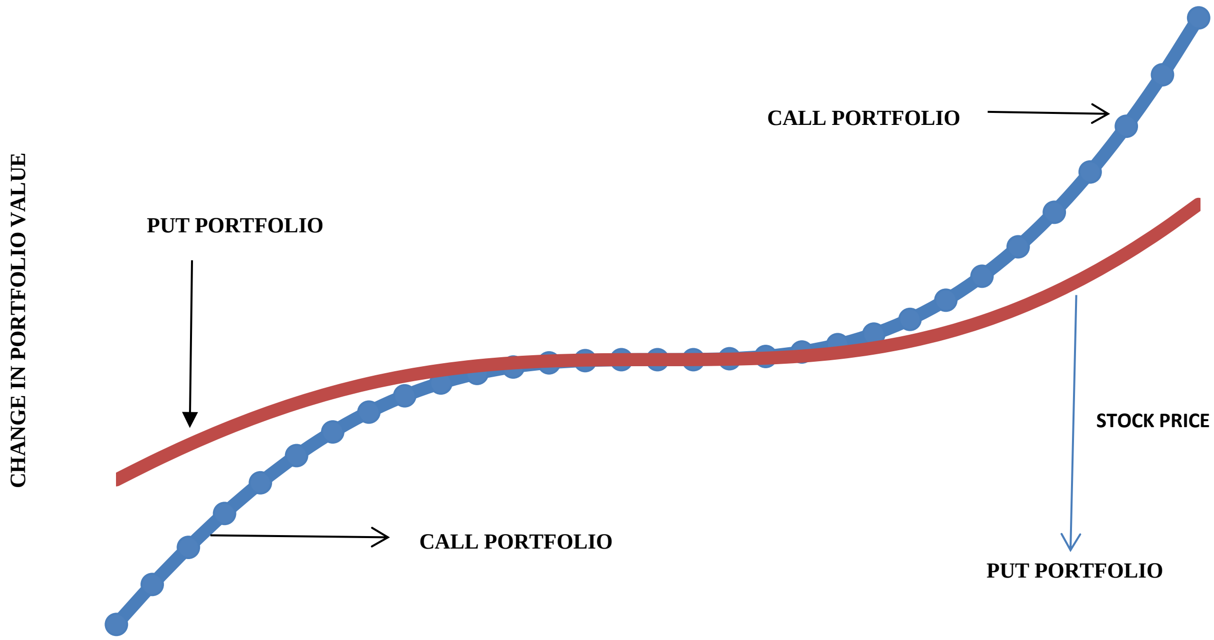

Table 4: Delta-Gamma Hedging with Calls

	OPTION 1	**OPTION 2**
CURRENT STOCK PRICE	$100.00	$100.00
EXERCISE PRICE	$100.00	$110.00
RISK-FREE RATE	5.00%	5.00%
TIME TO MATURITY (91 DAYS)	24.93%	24.93%
VOLATILITY	20.00%	20.00%
D1	0.1748	-0.7796
D2	0.0749	-0.8795
N(D1) [CALL DELTA]	0.5694	0.2178
N(D2)	0.5299	0.1896
PUT DELTA	-0.4306	-0.7822
CALL PRICE	$4.61	$1.19
PUT PRICE	$3.37	$9.82
CALL Z (d^2)	0.0153	0.3039
CALL / PUT GAMMA	0.039344	0.029479
INITIAL PORTFOLIO		
1000 LONG SHARES	1,000	
SELL 3588.2811 OF OPTION 1 CALLS	3,588	
BUY 4789.0392 OF OPTION 2 CALLS	4,789	
DELTA OF THE PORTFOLIO	0.00	
GAMMA OF THE PORTFOLIO	0.00	
THETA OF THE PORTFOLIO	4,457.37	
VALUE OF PORTFOLIO	$89,148	
BLACK-SCHOLES PORTFOLIO PDE	-0.002	

ENDING STOCK PRICE	**CALL VALUE OF OPTION 1**	**CALL VALUE OF OPTION 2**	**CHANGE IN THE VALUE OF THE DELTA NEUTRAL PORTFOLIO**
$100	$4.61	$1.19	$0
$85	$0.27	$0.02	-$4,998
$86	$0.35	$0.03	-$4,246
$87	$0.45	$0.04	-$3,547
$88	$0.57	$0.06	-$2,904
$89	$0.72	$0.08	-$2,324
$90	$0.89	$0.11	-$1,811
$91	$1.10	$0.15	-$1,365
$92	$1.34	$0.20	-$989
$93	$1.61	$0.26	-$682
$94	$1.92	$0.33	-$441
$95	$2.27	$0.42	-$261
$96	$2.65	$0.53	-$136
$97	$3.08	$0.66	-$59
$98	$3.55	$0.81	-$18
$99	$4.06	$0.98	-$2
$100	$4.61	$1.19	$0
$101	$5.20	$1.42	$2
$102	$5.82	$1.68	$18
$103	$6.49	$1.98	$61
$104	$7.19	$2.31	$145
$105	$7.92	$2.68	$282
$106	$8.68	$3.08	$484
$107	$9.47	$3.52	$762
$108	$10.28	$4.00	$1,124
$109	$11.12	$4.52	$1,578
$110	$11.98	$5.07	$2,130
$111	$12.86	$5.66	$2,785
$112	$13.76	$6.28	$3,544

where θ_p (theta of a call portfolio given by $\eta_c\ \theta$), Γ_p (gamma of a call portfolio given by $\eta_c\ \Gamma$) and C_p (Call/Stock portfolio given by: $\eta_s\ S + \eta_c\ C$). A delta hedged portfolio implies that $\Delta_p = 0$. Thus, (10) can be rewritten as:

$$r_f = \frac{\eta_c[\theta + (1/2)\Gamma\sigma^2 S^2]}{\eta_c[C + \frac{\eta_s}{\eta_c} S]} \qquad (11)$$

Delta hedging implies that for every share (η_s= 1), the number of calls (η_c) is given by the proportion -(1/Δ). Thus,

$$r_f = \frac{[\theta + (1/2)\Gamma\sigma^2 S^2]}{[C - S\Delta]} \qquad (12)$$

In essence, the equation above states that if each share of stock is delta hedged the portfolio is riskless and will hence earn the risk free rate of return. Using values from Table 1a it can be easily confirmed that the above is satisfied for calls and a similar relationship is satisfied for puts.

For delta-gamma neutral portfolios, $\Delta_P = \Gamma_P = 0$. Thus (10) reduces to:

$$r_f = [\theta_P / C_P] \qquad (13)$$

A delta/gamma neutral portfolio requires a short call position involving 3588 Option1 calls and a long position involving 4789 Option2 calls to hedge a 1000 share portfolio. The theta of this portfolio then equals θ_P = [(-3588)(-10.4852) + (4789)(-6.9255)] = 4455. The value of the call portfolio, C_P = [(-3588)($4.61) + (4789)($1.19)] = $89,158. The B-S PDE implies that a delta/gamma neutral portfolio is riskless and therefore earns the risk free rate of return.

Conclusion

The delta-gamma hedging concept is among the more challenging concepts in derivatives portfolio management. However, standard textbook exposition of delta-gamma hedging usually does not proceed beyond a perfunctory mathematical presentation of delta hedging with calls. Issues such as delta hedging with puts, contrasting delta hedging with calls against puts, gamma properties of call versus put delta hedges, etc. are usually not presented. The objective of this paper is to examine and illustrate these notions using simple stylized examples. These issues are then placed within the context of a fundamental result in derivatives theory - the Black-Scholes PDE. Concepts such as delta-gamma hedges and convexity of portfolio positions are presented using a simple diagrammatic framework. Hopefully, this approach complements the purely mathematical approach in many textbooks while clarifying and reinforcing the underlying intuition.

References

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