The Complete Set of Explicit Yield to Maturity Formulas

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Abstract

Obtaining general explicit yield to maturity (YTM) formulas is feasible for coupon bonds with integer maturities of less than five periods but, surprisingly, the literature lacks such formulas for three- and four-period bonds. I fill this void by providing the complete set of integer-maturity YTM formulas. Although the new formulas are unwieldy, they may be computationally advantageous over iterative methods of finding the YTM. Indeed, I present simulations indicating that the three-period YTM formula may require as little as one-third of the computation time needed with trial and error methods. Similarly, the four-period formula also produces substantial computational savings.

Introduction

The search for a coupon bond’s yield to maturity (YTM) has a long history in Financial Economics. However, most of the research on this topic focuses on finding approximate YTM formulas, given that it is impossible to obtain an explicit general formula for bonds with five or more periods to maturity.\(^2\) While explicit YTM formulas for integer bond maturities up to four periods are feasible, it is surprising that the YTM formulas for three- and four-period coupon bonds are not available in the literature. The purpose of this paper is to remedy this deficiency by providing the complete set of explicit integer-maturity YTM formulas.

The derivation of the explicit YTM formulas for three- and four-period bonds is far from trivial, requiring a substantial dose of ingenuity. Fortunately, this problem is equivalent to finding the roots of cubic and quartic polynomials, whose general solutions were found by mathematicians long ago. Therefore, to obtain the desired YTM formulas I apply these polynomial techniques to the specific case of coupon bonds. Although the three- and four-period YTM formulas are unwieldy and unintuitive, they may be programmed into software packages, permitting a direct calculation of the YTM as an alternative to trial and error methods.

Although the lack of intuition of the YTM formulas for three- and four-period bonds may help explain their otherwise baffling absence from the literature, their nonappearance may be unfortunate from a practical perspective because these YTM formulas, when applicable, may be computationally superior to trial and error methods. To test this possibility, I assess the practical value of all four explicit YTM expressions by comparing the time needed to compute a bond’s YTM using the formulas, against the computation time required by two popular trial and error methods. The simulations presented in this article indicate that the YTM formulas generally provide a significant computational advantage over both trial and error methods. To take an example, for some bond parameter combinations the three-period formula requires as little as one-third of the time consumed by trial and error methods.

The paper proceeds as follows. In the next section I give a brief restatement of the well known YTM formulas for one- and two-period coupon bonds, for the sake of completeness, and proceed to provide the new

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\(^2\) Hawawini and Vora (1982) provide an excellent historical account of approximate YTM formulas. Obtaining an explicit formula for the YTM of a five-period coupon bond is tantamount to finding a general explicit formula for the roots of a quintic polynomial, which was proven by Abel (1802-1829) to be impossible. For discussions on this topic see Bell (1965, pp. 307-326) and Gullberg (1997, pp. 297-301).
three- and four-period explicit formulas. Then I evaluate the computational time benefit of the four formulas against two popular trial and error methods. The final section summarizes.

**Explicit YTM Formulas for Integer-Maturity Coupon Bonds**

All the derivations that follow assume an integer-maturity coupon bond with current price \( P \), periodic coupon \( C \), and face value \( F \). Let \( c = C/P \) be the bond’s current yield, and let \( f = F/P \) be its face value per unit current price. The bond matures \( n \) periods from now, where \( n \in \{1, 2, 3, 4\} \). Then, the bond’s periodic YTM can be represented either in terms of the original set of parameters, \( \{n, C, F, P\} \), or in terms of the derived parameters, \( \{n, c, f\} \). Except for the one-period bond, the explicit formulas in this article are based on the derived parameters.

**One-Period Coupon Bonds**

In terms of the original set of bond parameters, the well known formula for one-period bonds is given by

\[
YTM_1 = \frac{C + (F - P)}{P}.
\]  

(1)

The intuition in equation (1) is straightforward, stating that the YTM of a one-period bond equals the sum of the periodic coupon and capital gain to be received by the bondholder one period from now, per dollar invested today.

In terms of the derived parameters, the explicit one-period formula is

\[
YTM_1 = c + f - 1.
\]  

(2)

Equivalently, for one-period coupon bonds the YTM equals the sum of the current yield, \( c \), and the capital gains yield, \( f - 1 \).

**Two-Period Coupon Bonds**

Finding the YTM formula for two-period coupon bonds requires solving the following quadratic equation, where \( R \equiv 1 + \text{YTM} \):

\[
R^2 - cr - (c + f) = 0.
\]  

(3)

Applying the quadratic solution for \( R \) in equation (3) gives the explicit YTM formula:

\[
YTM_2 = \frac{c}{2} + \left[ c \left( 1 + \frac{c}{4} \right) + f \right]^{1/2} - 1.
\]  

(4)

In contrast to the simple YTM formula for one-period bonds, equation (4) does not allow an intuitive interpretation. This trend of increasing complexity and decreasing intuition intensifies as the bond’s maturity increases to three and four periods. As stated previously, this absence of intuition may help explain the curious absence from the literature of explicit formulas for three- and four-period coupon bonds.

**Three-Period Coupon Bonds**

In contrast to the formulas for one- and two-period maturities, finding the explicit YTM formula for a three-period bond requires considerable effort and insight. The resulting formula is:

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3 The Appendix gives detailed derivations for the three- and four-period formulas.
\[ YTM_3 = \frac{c}{3} + \left[ -\frac{q}{2} + \left( \frac{g}{2} \right)^2 + \left( \frac{p}{3} \right)^3 \right]^{1/2} \right]^{1/3} + \left[ -\frac{q}{2} - \left( \frac{g}{2} \right)^2 + \left( \frac{p}{3} \right)^3 \right]^{1/2} \right]^{1/3} - 1, \] (5)

where \( p \) and \( q \) in equation (5) are defined as follows:

\[ p = -c(1 + c/3), \] (6)
\[ q = -c[1 + c/3 + (2/27)c^2] - f. \] (7)

Clearly, the computational effort required by the three-period YTM formula is substantial. Nevertheless, as already stated, this explicit formula may prove to be computationally advantageous. This empirical issue is evaluated in the next section.

**Four-Period Coupon Bonds**

Given the seemingly exponential increase in the complexity of the YTM formulas as the bond’s maturity increases, it should not be surprising to find that the explicit YTM formula for four-period bonds is even more convoluted than the three-period formula. The expression may be written as

\[ YTM_4 = \frac{c}{4} + \frac{1}{2} \left( p + 2z \right)^{1/2} + \frac{1}{2} \left( -3p - 2z + 4[(p + z)^2 - r]^{1/2} \right) \] (8)

where \( p, r, \) and \( z \) are defined as follows:

\[ p = -c(1 + 3c/8), \] (9)
\[ r = -c(1 + c/4 + c^2/16 + 3c^3/256) - f, \] (10)
\[ z = -\frac{5p}{6} + \left[ -\frac{k}{2} + \left( \frac{k}{2} \right)^2 + \left( \frac{j}{3} \right)^3 \right]^{1/2} \right]^{1/3} + \left[ -\frac{k}{2} - \left( \frac{k}{2} \right)^2 + \left( \frac{j}{3} \right)^3 \right]^{1/2} \right]^{1/3}. \] (11)

In equation (11), \( j \) and \( k \) are defined as

\[ j = -\left( r + \frac{p^2}{12} \right), \] (12)
\[ k = \left( \frac{p^3}{108} - \frac{rp}{3} + \frac{q^2}{8} \right), \] (13)

and \( q \) in equation (13) is given by

\[ q = -c(1 + c/2 + c^2/8). \] (14)

The four-period YTM formula, equations (8)-(14), is clearly formidable and, given its complexity, it seems reasonable to ask if it provides any computational benefit over the customary trial and error methods. This question is elucidated below.

**Computational Advantage of the Explicit YTM Formulas**

The complexity of the new formulas for three- and four-period coupon bonds presented above is undeniable, and the formulas lack intuitive content to boot. Despite these drawbacks, it is of practical importance to assess whether these equations reduce the computational burden of finding the YTM relative to trial and error methods. The answer to this question is by no means obvious, for two reasons. First, the explicit three- and four-period formulas are cumbersome and may require a relatively large amount of computing time. Second, the trial and error iterative methods may, in practice, converge rapidly to the exact YTM. Therefore, the computational advantage of the n-period formulas, \( n \in \{1, 2, 3, 4\} \), is an empirical question. In this section I present extensive simulations indicating that in most cases the explicit formulas compute the YTM of a coupon bond much faster than trial and error methods.
The trial and error methods require an initial YTM estimate, or guess, which may be provided by a member of the following one-parameter family of YTM approximations:

\[
YTM(\alpha) = \frac{C + \frac{F - P}{n}}{\frac{F + \alpha P}{\alpha + 1}}. \tag{15}
\]

The numerator of equation (15) can be interpreted as the periodic receipts from coupons and capital gains, and the denominator as the average price of the bond over its remaining life. Indeed, Rodriguez (1988) proves that if the n-period bond’s price at any time, \(t\), is given by \(P(t) = P + \frac{(F - P)/n^\alpha}{\alpha + 1}\), where \(t = 0\) is now and \(\alpha\) is a positive number, then the bond’s average price is given by the denominator in equation (15). Thus, \(\alpha = 1\) assumes that the bond follows a linear price path from \(P(0) = P\) to \(P(n) = F\); whereas \(\alpha = 2\) implies a quadratic price path, and so on. Notice that equation (15) gives the exact YTM in the special case of par bonds, \(P = F\), regardless of the values of \(n\) and \(\alpha\).

After studying the behavior of \(YTM(\alpha)\), Rodriguez (1988) concludes that the best yield to maturity approximations occur for \(\alpha = 2\) (the quadratic approximation), rather than for \(\alpha = 1\) (the linear approximation) or any other \(\alpha\) parameter value. Tavakkol (1995) reconfirms the general superiority of the quadratic approximation to the YTM. Nevertheless, given the ongoing popularity of the linear approximation, I compare the computational performance of the four explicit formulas against both the linear and quadratic trial and error methods.

The simulation procedure is the following. I create a computer program where the coupon rate, \(C/F\), and the YTM vary from 1 to 20 percent, in increments of 100 basis points. The program then calculates the price of the bond, \(P\), for each of these 400 parameter combinations, and uses this price along with the other parameters to obtain the bond’s YTM using the appropriate explicit formula. The program also calculates the YTM using the linear and quadratic trial and error methods described above. Finally, the computation time consumed by each of these three methods is recorded.

Table 1 summarizes the computation time needed to find the YTM for the 400 simulated bonds, using the explicit formulas and both trial and error methods. The average, minimum, and maximum computation times are reported for the three methods. The entries in panel A should be interpreted as relative time units because the actual computation time varies widely, depending on the hardware hosting the program. To take an example, the table indicates that the one-period formula requires an average of 5.7 time units per bond to compute the true YTM, and that the comparable figures for the linear and quadratic trial and error methods are 31.8 and 31.4 time units, respectively.

Panel B of Table 1 presents summary statistics for the ratio, in percent, of the computation time required by the explicit formulas, \(E\), to the time required by each of the trial and error methods, \(T\). Because the formulas become more complex as maturity increases, it is not surprising to find that the average \(E/T\) ratio also increases with the bond’s maturity. For example, for the linear trial and error method the average \(E/T\) ratio goes from 18.6 percent for \(n = 1\) period to 83.4 percent for \(n = 4\) periods. Thus, on average, all the explicit formulas are computationally faster than the linear trial and error method.

Panel B also shows that, for a given maturity, the \(E/T\) ratio is greater when \(T\) refers to the computation time used by the quadratic method. For example, for three-period bonds the average \(E/T\) ratio is 46.1 percent when \(T\) refers to the linear method, but it is 51.2 percent when referring to the quadratic method. This result is consistent with the observation that the quadratic estimate (\(\alpha = 2\)) is generally closer to the exact YTM than the linear estimate (\(\alpha = 1\)), resulting in a smaller convergence time.

Panel B indicates that the formulas for one- and two-period bonds compute the YTM faster than both trial and error methods, for all 400 combinations of bond parameters. This is seen by noting that the maximum value of the \(E/T\) ratio is less than 100 percent for these maturities. In contrast, the three- and four-period formulas are sometimes computationally slower than the trial and error methods. Closer inspection of the results, however, reveals that this behavior occurs only for bonds selling at, or near, par value. Intuitively, this happens due to the combination of two factors. First, the complexity of the explicit three- and four-period YTM formulas requires

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4 Notice that, by construction, each YTM computed is an integer percentage between 1 and 20, so its precise value is known. This design provides a simple way to verify the accuracy of the computed YTM to an arbitrary number of decimals.
Table 1. Computing Time Required by the Explicit and Approximate YTM Formulas

| Panel A: Relative Computing Time Needed With the Explicit Formulas and Two Trial and Error Methods |
|-------------------------------------------------|----------|----------|----------|----------|
| Explicit Formula                                | n = 1    | n = 2    | n = 3    | n = 4    |
| Average                                        | 5.7      | 7.3      | 14.5     | 26.7     |
| Minimum                                        | 4.7      | 6.6      | 13.7     | 25.8     |
| Maximum                                        | 7.0      | 8.2      | 16.0     | 28.1     |
| Trial & Error: Linear                          |          |          |          |          |
| Average                                        | 31.8     | 32.4     | 32.9     | 33.6     |
| Minimum                                        | 13.7     | 12.9     | 13.3     | 12.9     |
| Maximum                                        | 38.3     | 38.3     | 42.2     | 42.2     |
| Trial & Error: Quadratic                       |          |          |          |          |
| Average                                        | 31.4     | 31.0     | 29.4     | 29.5     |
| Minimum                                        | 13.7     | 13.7     | 13.7     | 13.7     |
| Maximum                                        | 38.3     | 37.1     | 37.1     | 36.3     |

Panel B: The E/T Percentage Ratio of the Computation Time with the Explicit Formulas (E) to the Computation Time with the Trial and Error Methods (T).

| Explicit versus Linear                         | n = 1    | n = 2    | n = 3    | n = 4    |
| Average                                        | 18.6     | 23.5     | 46.1     | 83.4     |
| Minimum                                        | 12.8     | 17.9     | 33.6     | 61.1     |
| Maximum                                        | 44.4     | 58.3     | 108.6    | 209.1    |
| Explicit versus Quadratic                     |          |          |          |          |
| Average                                        | 18.8     | 24.4     | 51.2     | 93.8     |
| Minimum                                        | 12.6     | 17.9     | 37.9     | 71.0     |
| Maximum                                        | 45.7     | 58.3     | 111.4    | 197.1    |

Notes: The entries in panel A depict the average, minimum, and maximum times required to compute the true YTM of an n-period coupon bond, using the explicit formulas given in the text as well as the linear and quadratic trial and error methods derived from equation (15), for 400 bond parameter combinations. These entries should be considered only as relative time units, given the great variety of possible hardware configurations that may host the computation software. Panel B shows the percentage E/T ratio, that is, the ratio of the computation time required to find the YTM with the explicit formulas (E), to the time required by each of the trial and error methods (T).

relatively large computation times and, second, for bond prices near par the YTM estimate from equation (15) is an excellent approximation to the exact YTM.

To explore in more detail the behavior of the explicit (E) and trial and error (T) computation times, Figure 1 shows the E/T ratio for 3-period coupon bonds with a YTM of 15 percent (graph A), and the E/T ratio for 4-period bonds with a YTM of 5 percent (graph B). In graph A the linear method is used to calculate T, and in graph B the quadratic method is used. For both graphs the coupon values are $1, $2, $3, …, $200. Over most of this range, the E/T ratio is essentially constant, although a spike occurs when the coupon rate equals the YTM; that is, when the bond sells at par. Notice that an increase in the E/T ratio also occurs in the proximity of the spike, reflecting the near-accuracy of the approximate YTM formula in this vicinity. The overall conclusion from Figure 1 is that the explicit YTM formulas are generally computationally superior to the trial and error methods, as the E/T ratio is almost always less than 100 percent.

5 Of course, the spike could be eliminated by calculating the YTM from equation (15) when the bond’s price equals its face value, rather than using the general explicit formulas.
Figure 1. Computational Advantage of the Explicit YTM Formulas Over Trial and Error Methods

A. Explicit Formula versus Linear Trial and Error Method; \( n = 3 \) periods, \( YTM = 15\% \)

B. Explicit Formula versus Quadratic Trial and Error Method; \( n = 4 \) periods, \( YTM = 5\% \)

Notes: The graphs depict the ratio of the computing time needed to obtain the true YTM of an \( n \)-period coupon bond when using the explicit formulas given in the text, to the time required with two trial and error methods derived from equation (15).

Summary

Although explicit YTM formulas for coupon bonds with integer maturities of less than 5 periods have long been known to be feasible, the three- and four-period expressions were unavailable in the literature. In this
article I remedy these surprising omissions by providing the complete set of explicit YTM formulas for integer-maturity bonds. Regrettably, the three-period formula is cumbersome, and the corresponding four-period expression is even more unwieldy.

While the complexity of the higher-maturity YTM formulas may help rationalize their prior unavailability, from a practical viewpoint their absence seems unjustifiable because the explicit YTM formulas, when applicable, may be computationally superior to trial and error methods. Indeed, the simulations presented in this article strongly indicate that this is almost always the case.

References


Appendix

Derivation of the Explicit Three- and Four-Period YTM Formulas

The Explicit Three-Period Formula

Recalling from the text that \( R = 1 + \text{YTM} \), \( c = C/P \) and \( f = F/P \), the basic valuation equation for three-period bonds is readily converted into the following cubic polynomial:

\[
R^3 - c R^2 - c R - (c + f) = 0. \tag{A.1}
\]

Let \( R = c/3 + x \), substitute it into equation (A.1), and simplify the resulting expression to get

\[
x^3 + p x + q = 0, \tag{A.2}
\]

where \( p \) and \( q \) are defined as follows:

\[
p = -c(1 + c/3), \tag{A.3}
\]

\[
q = -c(1 + c/3 + (2/27)c^2) - f. \tag{A.4}
\]

Equation (A.2) is known as the reduced form of equation (A.1), as it lacks the quadratic term. To solve it, assume the cubic equation (A.2) has a root equal to \( x = u + v \). Substituting this root into equation (A.2) and simplifying gives

\[
(u^3 + v^3) + (3uv + p)(u + v) + q = 0. \tag{A.5}
\]

Equation (A.5) will be satisfied if \( 3uv = -p \) and \( u^3 + v^3 = -q \); equivalently, if \( u v^3 = (-p/3)^3 \) and \( u^3 + v^3 = -q \). Now notice that these last two conditions jointly imply that \( u^3 \) and \( v^3 \) are the solutions to the quadratic equation \( w^2 + q w + (-p/3)^3 = 0 \). Therefore, the expressions for \( u^3 \) and \( v^3 \) can be found by solving for the two roots of this quadratic equation. Thus, \( u \) and \( v \) are given by

\[
6 \text{ To see this, recall that if a quadratic equation } x^2 + bx + c = 0 \text{ has roots } g \text{ and } h, \text{ it can be written as } (x - g)(x - h) = 0, \text{ or } x^2 - (g + h)x + gh = 0. \text{ Thus, } gh = c \text{ and } g + h = -b.\]
\[ u = \left[ -\frac{q}{2} + \left\{ \frac{q}{2}^2 + \left( \frac{p}{3} \right)^3 \right\}^{1/2} \right]^{1/3}, \]  
\[ v = \left[ -\frac{q}{2} - \left\{ \frac{q}{2}^2 + \left( \frac{p}{3} \right)^3 \right\}^{1/2} \right]^{1/3}. \]  

Since \( x = u + v \), \( R = c/3 + x \), and \( \text{YTM} = R - 1 \), we finally have

\[ \text{YTM}_3 = \frac{c}{3} + \left[ -\frac{q}{2} + \left\{ \frac{q}{2}^2 + \left( \frac{p}{3} \right)^3 \right\}^{1/2} \right]^{1/3} + \left[ -\frac{q}{2} - \left\{ \frac{q}{2}^2 + \left( \frac{p}{3} \right)^3 \right\}^{1/2} \right]^{1/3} - 1, \]

where \( p \) and \( q \) are defined in equations (A.3) and (A.4), respectively. Equation (A.8) is the same as equation (5) in the text.

The Explicit Four-Period Formula

Define \( R = 1 + \text{YTM} \) as before. Then the YTM of a four-period bond obeys the following implicit relationship:

\[ R^4 - cR^3 - cR^2 - cR - (c + f) = 0. \]  

Let \( R = c/4 + x \) and substitute it into equation (A.9). After much simplification we obtain

\[ x^4 + px^2 + qx + r = 0, \]

where \( p \), \( q \), and \( r \) are defined as follows:

\[ p = -c(1 + 3c/8), \]  
\[ q = -c(1 + c/2 + c^2/8), \]  
\[ r = -c(1 + c/4 + c^2/16 + 3c^3/256) - f. \]

The strategy for solving the quartic equation (A.10) is to reduce it to a cubic equation, which can then be solved using the methodology for three-period bonds. Reducing a quartic to a cubic requires the ingenious step of rewriting equation (A.10) as follows:

\[ (x^2 + p + z)^2 = (p + 2z)x^2 - qx + (p^2 - r + 2pz + z^2). \]

Since the left-hand side of equation (A.14) is a perfect square, the right-hand side must be a perfect square too. It follows that the discriminant of the right-hand side equals zero.\(^7\) This condition leads to

\[ 4(p + 2z)(p^2 - r + 2pz + z^2) = q^2. \]

Multiplying and rearranging equation (A.15) gives the following cubic equation in \( z \):

\[ z^3 + (5/2)pz^2 + (2p^2 - r)z + (p^3 - pr - q^2/4)/2 = 0. \]

This cubic is called the resolvent equation, because its solution allows us to find the roots of the quartic which, in turn, leads to the explicit YTM formula for a four-period bond. Let \( z = y - (5/6)p \), substitute it in equation (A.16) and simplify to get the reduced equation

\[ y^3 + jy + k = 0, \]

where \( j \) and \( k \) are defined as follows:

\[ j = \left( -r + \frac{p^2}{12} \right), \]  
\[ k = \frac{2pq}{12}. \]

\(^7\) If a quadratic equation \( ax^2 + bx + c \) is a perfect square, it can be written as \((mx + n)^2\) for some \( m \) and \( n \). It is readily seen that \( m^2 = a, n^2 = c, \) and \( 2mn = b \). These last three expressions imply that \( b^2 - 4ac = 0 \).
Since equation (A.17) is a cubic with no quadratic term, like equation (A.2), it can be solved similarly. Recalling that \( z = y - (5/6)p \), a straightforward adaptation of the solution to the cubic gives

\[
\begin{align*}
    z &= -\frac{5p}{6} + \left( -\frac{k}{2} + \left[ \left( \frac{k}{2} \right)^2 + \left( \frac{j}{3} \right)^3 \right]^{1/2} \right) + \left[ -\frac{k}{2} \left( \left( \frac{k}{2} \right)^2 + \left( \frac{j}{3} \right)^3 \right)^{1/2} \right]^{1/3},
\end{align*}
\]

(A.20)

where \( p, j, \) and \( k \) were defined in equations (A.11), (A.18), and (A.19), respectively.

Now that we have an expression for \( z \), we again turn our attention to solving equation (A.14) for \( x \), as this is required to find the YTM formula. Since the right-hand side of equation (A.14) must be a perfect square, it can be written as

\[
(p + 2z)x^2 - q x + (p^2 - r + 2pz + z^2) = (mx + n)^2.
\]

(A.21)

Thus, equation (A.14) may be rewritten as

\[
(x^2 + p + z)^2 = (m x + n)^2.
\]

(A.22)

Solving equation (A.21) for \( m \) and \( n \) gives

\[
\begin{align*}
    m &= (p + 2z)^{1/2}, \\
    n &= (p^2 - r + 2pz + z^2)^{1/2}.
\end{align*}
\]

(A.23) \hspace{1cm} (A.24)

Substituting these expressions into equation (A.22), taking square roots on both sides, and rearranging, gives a quadratic equation in \( x \):

\[
x^2 - (p + 2z)^{1/2} x + [p + z - (p^2 - r + 2pz + z^2)]^{1/2} = 0.
\]

(A.25)

Notice that, after much effort, the quartic equation (A.10) has been reduced to a quadratic equation. Solving equation (A.25) for \( x \), and recalling that \( 1 + \text{YTM} \equiv R \equiv x + c/4 \), we arrive at the long-sought formula for the YTM of four-period bonds,

\[
\text{YTM}_4 = \frac{c}{4} + \frac{1}{2} (p + 2z)^{1/2} + \frac{1}{2} \left[ -3p - 2z + 4[(p + z)^2 - r]^{1/2} \right]^{1/2} - 1,
\]

(A.26)

where \( p, q, r, j, k, \) and \( z \) are given in equations (A.11), (A.12), (A.13), (A.18), (A.19), and (A.20), respectively. Equation (A.26) is the same as equation (8) in the text.