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Evaluating an Online Capital Budgeting Simulation Game in an MBA Financial Management Course

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ABSTRACT

In this paper, the instructor introduces and evaluates an online capital budgeting simulation game in an MBA financial management course at an AACSB accredited school. A survey method is utilized to assess students' opinions about the game. The survey results indicate that the majority of students felt that the simulation game helped them learn financial management knowledge better, preferred this game approach, and would recommend it for future finance courses. A statistical analysis is conducted to assess the effectiveness of the simulation game on students' learning. The statistical result indicates that the simulation game significantly improved students' learning outcomes.

Introduction

Capital budgeting involves a firm's decision to allocate scarce capital to long-term assets efficiently. Capital budgeting is one of the essential functions of financial management in business organizations, and its importance keeps growing in recent years. However, it is difficult to teach this concept to business students using traditional teaching methods, such as lecturing or case studies. It is also challenging to teach MBA students who have no or limited business experience. The difficulty of using traditional teaching methods may be due to the following two reasons. First, in today's dynamic business world, projects are often interdependent, involve non-traditional cash flows, and include real options. Second, since the investment decisions that a manager makes in this year affect his or her investment opportunity set in future years, intertemporal trade-offs should be considered. An online simulation game may be a great option to overcome this difficulty because it can simulate the dynamic business environment and enable students to confront intertemporal trade-offs to make appropriate funding decisions.

An online simulation game is typically a computer-oriented program that enables students to manage a virtual company and make business decisions in a dynamic environment. Compared to traditional lecture-based teaching methods, a simulation game may help students apply business principles, concepts, and theories to practice, make the learning process exciting and interactive, and provide experiential learning environments and scenarios that would otherwise be impossible or infeasible for learners to encounter.

In this paper, the instructor introduced and evaluated an online capital budgeting simulation game in an MBA financial management course. This game was developed and operated by Harvard Business Publishing (HBP 2010). In this game, students play the role of CEO of a doll company in selecting projects and allocating capital across the company's divisions under a budget constraint. Not only does the game help students develop a set of capital budgeting related managerial skills, but it also provides broad knowledge and information that relates to other financial concepts and theory, such as financial statements, time value of money, cash flow estimation, risk and return, etc. The online game approach is expected to enhance and strengthen students' learning in financial management as well.

This paper's objective is to evaluate the online finance simulation game and assess its effectiveness on students' learning in an MBA financial management course. A survey was used to evaluate and analyze students' opinions about their learning from the game. Statistical analysis was conducted to assess the effectiveness of the simulation game on students' financial management learning through a comparison between classes using the online game and classes with a traditional teaching approach. The simulation was also intended to introduce the game approach to other finance-related courses.

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The rest of the paper is organized as follows. The next section briefly reviews the literature. The third section describes the capital budgeting simulation game. The fourth section introduces the teaching processes of the simulation game. The fifth section discusses the game survey and statistical analysis. Finally, the sixth section concludes the study.

Literature Review

With the advent of computer and internet technologies, simulation games have become a popular pedagogical tool for teaching concepts in different business areas over the past two decades. For instance, in the marketing area, Gillentine and Schulz (2001) used a fantasy football league simulation game to teach sports marketing concepts. In the management area, Shannon et al. (2010) adopted a simulation game for teaching lean manufacturing implementation strategies. In operations management, Feng and Ma (2008) reviewed and evaluated an online simulation game for teaching supply chain management concepts. Likewise, Pasin and Giroux (2011) presented a new simulation game and analyzed its impact on operations management education. In finance, simulation games have been adopted for teaching a variety of concepts such as investment, money and banking, and personal finance. McClatchey and Kuhlemeyer (2000) reported that over 70% of finance professors surveyed used different kinds of simulation exercises or games in their investment courses. Santos (2002) developed an internet-based financial system simulator to teach students the domestic and international consequences of monetary policy for an undergraduate money and banking class. Pavlik and Nienhaus (2004) presented a structured real-time classroom options trading game for an undergraduate investment class. Jankowski and Shank (2010) analyzed a sample of online simulators to determine the suitability of each for various courses that teach stock, bond, and derivatives investment. Huang and Hsu (2011) explored the use of online games to teach personal finance concepts at the college level and concluded that integrating online games into coursework significantly enhanced students' learning outcomes.

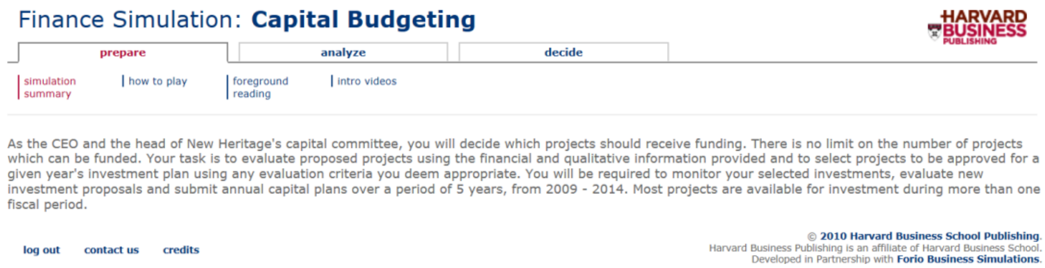
Although a variety of studies have been conducted to evaluate simulation games in the finance field, little research has been performed to study simulation games in capital budgeting, which is one of the most important building blocks of financial management. To the best of our knowledge, the pedagogical effectiveness of Finance Simulation: Capital Budgeting, which was developed by HBP in 2010, has never been evaluated and analyzed in any research articles. The game is designed to teach students capital budgeting concepts by having them analyze and select various projects under financial budget constraints and in a dynamic international setting. The game is entirely online and played in real time. In this paper, the instructor explores the use of the HBP finance simulation game in a financial management course and aims to bridge the research gap in evaluating online capital budgeting simulation games.

Description of the Capital Budgeting Simulation Game

The Finance Simulation: Capital Budgeting game is an internet-based capital budgeting simulator. The simulation is a single-player game and not designed to play against the computer, other students, or student teams. During the simulation game, students play the role of CEO and the head of the capital committee of New Heritage Doll Company in selecting projects and allocating capital across the company's divisions. New Heritage has three operating divisions: a doll and doll-accessory production division, a retailing division, and a licensing division. The production division designs and assembles dolls, doll accessories, and children's accessories into the finished product and then packages them for shipment. The retail division manages the sale of the dolls and accessories that the production division produces. The licensing division licenses the rights to New Heritage's branded doll characters and storylines to media publishing companies for use in books, software, movies, and other products. Students need to analyze and evaluate 27 competing investment projects and make funding decisions for those projects among the three divisions over a five-year period. The competing investment projects include replacement investments, expansion investments, investments in mutually exclusive projects, interdependent projects, and projects with growth options. To analyze and evaluate the projects, students should understand the project description, examine the cash flow patterns, consider the project budget constraints, and calculate standard metrics such as net present value (NPV), internal rate of return (IRR), profitability index (PI), payback period, etc. The goal of the game is to generate financial growth for the company through project evaluation and selection based on the given financial and qualitative information.

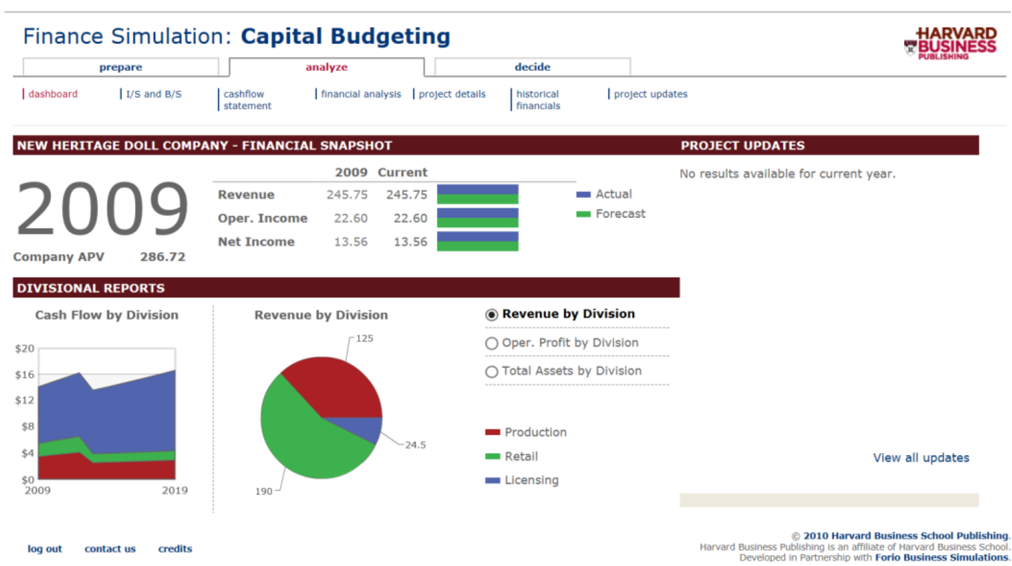
Playing the game consists of three steps. First, students review the **Prepare** tab, where they can read a summary of the simulation and review the foreground information about the company's three divisions and its corporate strategy. A short introduction video of the game is also available in this step to familiarize students with the basics of the game. Figure 1 shows a sample screen of the first step.

Figure 1: Prepare Step of the Game



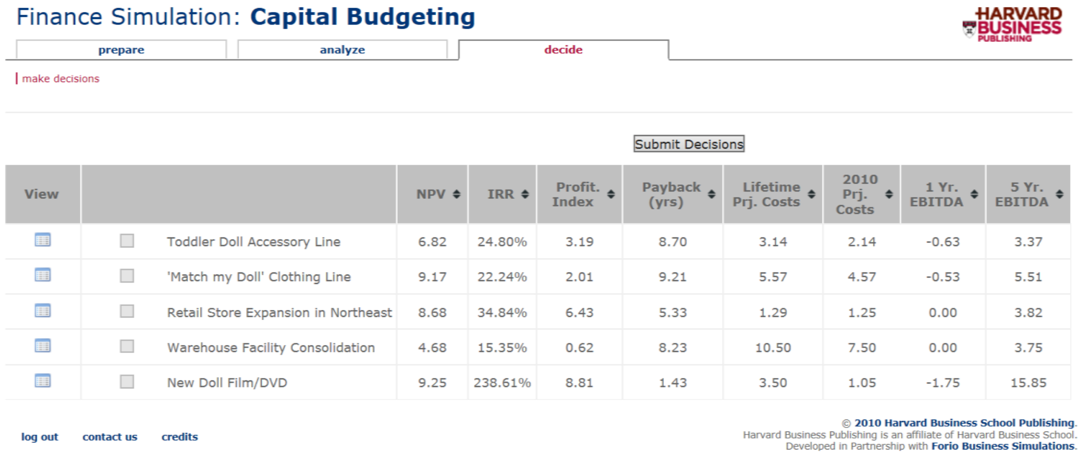
Second, students move to the **Analyze** tab. Here, students start to familiarize themselves with the dashboard containing an overview of the company's financial status. In this tab, students can review the financial snapshot of the company, divisional reports, project details, project updates, annual budget, and the detailed financial reports, such as income statement and balance sheet, cash flow statement, and financial analysis. Students can select the potential projects and preview how their selection will affect each budget sector through highlighted updates in the financial reports. Figure 2 contains a sample screen of the second step.

Figure 2: Analyze Step of the Game



Third, after analyzing the projects, students move on the **Decide** tab. The game starts with five projects, but more projects become available as the simulation progresses. As shown in Figure 3, students can sort the projects based on factors such as NPV, IRR, PI, and payback period in years. Students can also review the details of each project at this tab. When students make up their minds, they can choose those projects and submit their decisions. If the project selection exceeds the available budget, students must reselect the projects and make sure that the budget constraints are met. After students submit their decisions, the simulation advances to the next year, and the financial results of students' investment choices are available for them to review. Students can review the financial results and then proceed to evaluate and select projects for the subsequent four years of the game.

Figure 3: Decide Step of the Game



View			NPV	IRR	Profit. Index	Payback (yrs)	Lifetime Prj. Costs	2010 Prj. Costs	1 Yr. EBITDA	5 Yr. EBITDA
		Toddler Doll Accessory Line	6.82	24.80%	3.19	8.70	3.14	2.14	-0.63	3.37
		'Match my Doll' Clothing Line	9.17	22.24%	2.01	9.21	5.57	4.57	-0.53	5.51
		Retail Store Expansion in Northeast	8.68	34.84%	6.43	5.33	1.29	1.25	0.00	3.82
		Warehouse Facility Consolidation	4.68	15.35%	0.62	8.23	10.50	7.50	0.00	3.75
		New Doll Film/DVD	9.25	238.61%	8.81	1.43	3.50	1.05	-1.75	15.85

Adoption and Teaching of the Simulation Game

The simulation game was adopted and incorporated into a financial management course in an AACSB-accredited business school at a southeastern public university. The financial management course is a required core class of MBA program, and this class requires a prerequisite finance foundations course or its undergraduate equivalent. All students enrolled in this course are first-year MBA students, most of whom have working experience but have little or no experience playing online simulation games. Some of the students have undergraduate business degrees, but some do not have a business background at all. Those non-business students had to pass one prerequisite introductory finance course before they could register the financial management course. Hence, all students in the financial management course should have a basic understanding of general finance concepts.

Not only can the simulation game be used in an MBA financial management course, but it can also be adopted in undergraduate finance courses and executive education programs. According to the simulation game's website, it is appropriate for introductory finance courses and specialized courses such as project finance, capital budgeting, advanced corporate finance, and accounting. The game is also suitable for strategy and general management courses in which the topic of resource allocation is explored.

The detailed adoption process is as follows. First, the instructor visits the game's website. The website briefly introduces the simulation game and its learning objectives. The instructor must register an educator account to have access to a free trial, watch the introductory videos, and obtain teaching notes. Second, after a satisfactory free trial, the instructor may adopt this game and add it to a course-pack. Then, the instructor needs to activate the game and set up scenarios, budget constraints, and other parameters for the game. Third, after the initial game setup, the instructor can send the link of the course-pack to all the students in class for sign-up and purchase. Once students have purchased the game, the instructor can assign a game scenario to students. Then, students can start to play the game individually.

The teaching process of the game can be organized into three phases: pre-game, in-game, and post-game. Before students play the simulation game, the instructor covered the basic capital budgeting concepts related to the game in class. Those concepts included, but were not limited to, time value of money, mechanics of discounting, cash flow projection, business valuation, cost of capital, and four basic evaluation criteria for capital budgeting (NPV, IRR, payback period, and PI). For each evaluation criterion, the instructor used an example to illustrate the concept, calculation, evaluation process and rule, and the final decision. The instructor also discussed the advantages and disadvantages of each evaluation criterion. The instructor explained the impact of project interdependence, capital rationing, projects with non-traditional cash flows, and the concept of real options as well. Before the game started, the instructor also assigned students to read the background information about the game outside of the class and briefly introduced the features of the game in class.

Students play the simulation game online outside of class and it typically takes 30 to 60 minutes to run. Students were assigned a hybrid budget with both exogenous and endogenous parameters – a fixed dollar amount plus a percentage of the prior year's EBITDA ($5 million fixed amount each year plus 25% of previous year's EBITDA). Students had access to the project information, cash flows, and financial statements information. Students were not allowed to change the project discount rate.

After the game was finished, students wrote an individual report about the simulation game to explain why their decisions were made. They submitted the report, and its grade was part of their simulation project grade. The instructor analyzed students' decisions and outcomes and offered a debriefing session in class. During the debriefing session, the instructor first examined and ranked students' performance regarding the adjusted present value (APV) of the company, the operating cash flow during the five years of play, and other metrics, such as ending sales, EBIT, net income, and total assets. Then, the instructor examined each student's investment decisions and briefly presented the distribution of their decisions for a given period of the game. The instructor also identified the general pattern of students' decisions and common mistakes that students made. Finally, the instructor summarized the critical learning points for the simulation game and presented the linkage between this game and financial management concepts. A sample concept linkage is summarized in Table 1.

Table 1: Linkage between Capital Budgeting and Other Finance Concepts and Method

Game Attribute	Financial Management Concepts and Methods
Average Accounting Returns	Financial Statements, Cash Flows
NPV, IRR	Time Value of Money, Discounted Cash Flow Valuation
Payback Period	Cash Flow Estimation
NPV, IRR, PI	Cost of Capital, WACC, Risk and Risk-adjusted Discount Rates
Adjusted Present Value (APV)	Cost of Capital, WACC, Hurdle Rate

Game Survey and Statistical Analysis

To assess the effectiveness of using the online simulation game in the financial management course, the instructor designed and conducted a student survey. Based on Ruohomäki (1995) and Feng and Ma (2008), the instructor designed nine survey questions to evaluate the impact of simulation game on teaching financial management. The first six questions are multiple-choice questions on a five-point Likert scale, with five indicating "strongly agree" and one indicating "strongly disagree." The other three questions are open-ended questions to understand some factors that students like or dislike about the simulation game and any suggestions for improvement. Out of the multiple-choice questions, the first four questions focused on evaluating the effects on individuals, such as cognitive learning outcomes and the impacts of the game on participants' attitudes. The other two multiple-choice questions were designed to examine the efficacy of the simulation game as a teaching aid to the curriculum and future course.

According to students' responses (n = 23), the instructor computed the average score for each question; the results are listed in Table 2. Despite the class size being relatively small, the response rate of the survey is close to 100%. The results show positive impacts of the online simulation game on both the individual level and overall course level. The average of the first four questions is 4.385, and the average of questions 5 and 6 is 4.38. The survey results indicate that most of the students thought the simulation game helped them learn finance, actively thought about the simulation game, and enjoyed the game. The survey also confirms that students like this new teaching method and would recommend it for future finance courses.

Table 2: Student Survey on the Online Simulation Game

Question Number	Survey Questions	Average Score
1	Simulation game helped me in understanding basic concepts in Capital Budgeting	4.31
2	The game increased my interest and knowledge about finance	4.31
3	I frequently found myself actively thinking about the simulation game and what decisions I should make	4.46
4	I enjoyed playing the game	4.46
5	The game has a positive contribution to the curriculum	4.38
6	I recommend the game for future finance courses.	4.38
	*The **average** score of the all above questions*	4.38

Besides the multiple-choice questions, the survey also asked students three open-ended questions. Selected responses to those questions are listed in Table 3.

Table 3: Open-ended Survey Questions

Question Number	Survey Questions
7	List below the factors you liked about the simulation game. Selected Comments: • *"Some of the factors I liked about the simulation is it allowed me to gain a positive understanding of basic concepts in capital budgeting. It also allowed me to understand course from a financial aspect."* • *"Learned capital budgeting in a fun way."* • *"Each student has leadership from the game and can control business."* • *"I feel like I understand capital budgeting better."* • *"I received experience in making financial decisions."* • *"Some of the factors that I enjoyed about the simulation game is that it gave us real-life scenarios to make us think strategically to resolve each issue."* • *"I liked the game give actual project detail and financials. The simulation game helps to make individual to understand finance better."* • *"I like a hands-on approach on actually making decisions for a company. I feel like this is an effective way to learn besides the standard way of just reading the material."*
8	List below the factors you disliked about the simulation game. Selected comments: • *"Disliked the fact that we could not control discount rate."* • *"Complex income statements."* • *"Many of the numbers I did not understand and had to look up many of the terms."* • *"I disliked that we have to purchase the game."*
9	List below any suggestions you have on the simulation game for improvements. Selected comments: • *"Further explanation from the video."* • *"I suggest more videos on how to navigate the game and more examples."* • *"Provide equations for capital budgeting under preparation section。"*

To assess the game's impact on students' learning, the instructor compared the students' performance between Group A, taught without the game, and Group B, taught with the game. Each group consists of two classes from different semesters in recent years. The same instructor taught all classes and used the same textbook. Group A has 13 students from Spring 2013 and Spring 2014 classes, and students in this group received traditional lectures and homework assignments. Group B has 23 students from Spring 2016 and Spring 2017 classes, which assigned the simulation game as a project. The students in Group A were required to turn in homework assignments in paper format, whereas the students in Group B were requested first to play the game and then turn in a written summary report and the game survey form. For both groups, lectures on capital budgeting were delivered before the midterm exam. For Group B, students were requested to play the game after the midterm exam and submit the game-related assignment two weeks before the final exam. Table 4 reports the learning outcomes for both groups. As shown in Table 4, the average midterm score for Group A was 70.75 with a standard deviation of 8.37, as compared to an average score of 67.96 with a standard deviation of 12.84 for Group B. The average final exam score for Group A is 78.17 with a standard deviation of 9.10, as compared to an average score of 86.37 with a standard deviation of 9.79 for Group B. The instructor conducted t-tests to test the difference in exam scores between groups. The instructor found that the difference of final exam scores between groups is statistically significant at the 0.05 level of significance (p-value=0.028), whereas the difference of midterm exam scores between groups is not statistically significant. This demonstrates that students' learning outcomes are about the same between groups before the instructor introduced the simulation game. After the simulation game was introduced, the learning outcomes of Group B were significantly better than those of Group A.

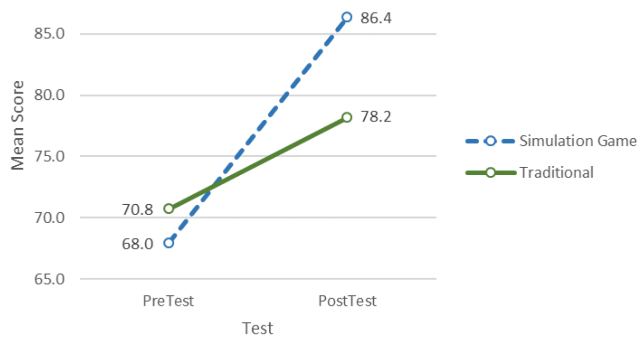
Table 4: The *t*-Test for Student Learning Outcomes

	Group A (Traditional)	Group B (Simulation Game)	*t*-test (*p*-value)
Midterm (Pre-test)	70.75	67.96	1.61
	[8.37]	[12.84]	(0.12)
Final Exam (Post-test)	78.17	86.37	-1.98
	[9.10]	[9.79]	(0.028*)
Difference	7.42	18.41	
Paired *t*-test	-2.08	-6.57	
(*p*-value)	(0.064)	(0.00*)	

The average test scores for both groups are reported. There were 13 students in Group A and 23 students in Group B. Standard deviations on average scores are reported in brackets and *p*-value of *t*-test are listed in parentheses. *Significant at the .05 level

To examine whether individual students on average experienced an improvement in test scores, the instructor also conducted t-tests to test the difference between midterm exam score and final exam score within groups. The statistical result showed that the difference in exam scores within Group B is statistically significant at 0.05 level of significance, whereas the difference in exam scores within Group A is not statistically significant. This indicated that after the simulation game was introduced, the learning outcomes of Group B significantly improved, while those of Group A did not improve. Figure 4 shows the comparison of mean test scores between the two groups. This result indicates that the simulation game significantly enhanced students' academic performance.

Figure 4: Comparison of Mean Scores (Pre-test vs. post-test)



A one-way analysis of covariance (ANCOVA), which controls the pre-existing differences between different intervention groups, was performed on students' final exam scores. The results are shown in Table 5. A statistically significant difference was found between the mean final exam scores of students who were exposed to the simulation game and those who were not exposed to the simulation game (*F* statistic = 6.13 with *p*-value = .02). This result indicates that students who played the simulation game had a statistically significant higher final exam score than those who did not play the game. This difference also demonstrates the simulation game has a positive impact on students' learning outcomes.

Table 5: Analysis of Covariance (ANCOVA) Results / Students Final Exam (Post-test) scores

Source of Variance	Sum of Squares	Def	Mean Squares	*F statistic*	*p*-value
Group	570.92	1	570.92	6.13*	0.02*
Midterm (Pre-test)	40.82	1	40.82	0.44	0.51
Error	2979.95	32	93.12		
Total	3551.39	34			

Pretest scores are the covariates. * Significant at the .05 level

Conclusion

In this paper, the instructor implemented and evaluated an online finance simulation game. As a complement to traditional teaching approaches, the simulation game is a useful teaching tool for teaching financial management. The simulation game allows students to apply knowledge learned in lectures in a dynamic experiential learning environment that would otherwise be impossible or infeasible for students to encounter. Students have hands-on opportunities to analyze the situation, make decisions, and observe the impact of their decisions in the game.

The research findings indicated that the online simulation game significantly improved students' learning outcomes. As a result, student test scores have dramatically improved after the simulation game was introduced. The results of the game survey indicate that most of the students thought the simulation game helped them learn knowledge about both capital budgeting and financial management, actively thought about the simulation game, and enjoyed the game. The survey also confirms that students like this new teaching method and would recommend it for future finance courses.

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Teaching the Quantity Theory of Money: A Simple Classroom Game

Stephen Norman and Douglas Wills[1]

ABSTRACT

This paper presents a classroom exercise that helps students understand the quantity theory of money and the implied relationship between money supply growth and inflation. In addition, the role and meaning of velocity is highlighted. The activity is easily implemented and only requires paper, pencil, and a spreadsheet to record data generated by the students. One major advantage of this game is that the outcome will almost always support the theoretical implications of the quantity equation. This is opposed to other simulations, which can sometimes differ dramatically from the model's prediction.

Introduction

Milton Friedman's assertion, "Inflation is always and everywhere a monetary phenomenon," is one of the best-known quotations about macroeconomics. All macroeconomists, as well as many in the general public, are familiar with this saying. Yet to the typical student, the statement is incomprehensible. Of those students who still read newspapers, some think inflation is caused by the rise in the price of specific goods. Fewer still have any understanding of concepts such as monetary aggregates.

Central to understanding Friedman's claim is comprehension of the quantity equation of money.[2] While not all macroeconomists consider the quantity equation useful for thinking about inflation, it is crucial for understanding monetarism, market monetarism, and the role of central banks (see Friedman 1983, Sumner 2015, and Christensen 2011). Regardless of one's views, we argue the quantity equation should be discussed in every macroeconomics course as the starting point for discussing inflation and the important role theory plays in understanding economic phenomena.[3]

To that end, we introduce an in-class experiment or game that introduces the quantity equation, exploits the fact that it is an identity (yet still useful for understanding inflation), and demonstrates the relationship between macroeconomic aggregates. An additional benefit is that it helps students understand the highly abstract and difficult to understand concept of the velocity of money.

Games or experiments are now commonplace in classrooms in part because they help students understand difficult topics. They are also highly effective ways to demonstrate the implications of economic models that students can observe from the results of their participation in the simulations. Despite the large literature on classroom games, there is a paucity of games based upon inflation and the quantity equation of money. In a survey of non-computerized games for both microeconomics and macroeconomics, only two games deal with the principle of inflation (Brauer and Delemeester 2001). This paper addresses that oversight.

Quantity Equation of Money

The quantity equation used for the game is:

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[2] Note there is distinction between the equation of exchange, which is defined as MV=PY, and the quantity theory of money, which assumes that the Y and V are constant. We thank an anonymous referee for pointing this out.

[3] If macroeconomics courses only included material that all economists agreed on, then such courses would be very short indeed.

$$MV=PY,$$

where M is the money stock, V is velocity, P is the average price, and Y is the number of goods sold.[4] As such, PY is equal to nominal output (or nominal GDP). From this it is straightforward to show that $\%\Delta M + \%\Delta V \approx \%\Delta P + \%\Delta Y$, and that the latter two terms are inflation and the real growth rate. This links the quantity equation to well-known terms and provides the basis for discussion after the game is completed.

In this game, the instructor determines the money stock, M, and output, Y. By defining velocity as

$$V = \frac{PY}{M},$$

we turn the equation into an identity. As described below, the students implicitly determine V by how much money they spend of the total amount they are allocated. Thus, when the instructor changes M or Y, the students, by their actions in the game, affect the price level.

The game uses the familiar setup used in market simulations that demonstrates how the standard supply and demand model can predict the outcome of the interaction of students acting as buyers and sellers. In this simulation, the market is "open outcry," allowing buyers and sellers to engage in mutually beneficial exchange of a good with little or no direction from the instructor.

The crucial difference in this game is that buyers are given a stock of money (as opposed to an imputed marginal value) to spend and the sellers have a stock of goods to sell (as opposed to an imputed marginal cost). The goal of buyers is to buy as many goods as they can with the money they are given, and the goal of sellers is to sell their endowed stock of goods for as much money as they can. As in the typical supply and demand games, the predictive power of the game played in such a chaotic manner makes the results more impressive to students. In the former game, students are unfailingly impressed that the instructor knows the average price that will emerge before the game begins. Similarly, students are impressed with how well the quantity equation describes the results of separate rounds of uncoordinated interactions of students. These experiments bring credibility to these models that is not possible via standard "chalk and talk" methods.

Game Setup

The inflation game is very simple to implement. Students play the game in rounds and only use a single card[5] per round and a sheet of paper to record their actions. The instructor uses only a computer with a spreadsheet and projector to display the results to the class. The novel aspect of this game is that the outcome of the game will almost always support the implications of the quantity equation of money. This is based on the fact the number of goods is constant in each round and the demand for money should also remain constant. This then implies that the growth in the money supply should be equal to the growth in the price level.

To begin the game, first physically divide the class into two groups, buyers and sellers, with the only instruction being that each student brings something with which to write. Each member of a group interacts only with members of the other group.[6] It is rare, but at times students do end up making transactions with members of their same group. In other words, a buyer purchases a good from another buyer. This usually happens when all they are doing is yelling out a price without saying whether they are buying or selling. One way to ensure that this type of mistake doesn't happen is to make the cards with their endowments different colors depending on whether they are a buyer or a seller. The instructor can point out that when they make a transaction, they should ensure the other individual has a card with a different color.

As a practical matter, it is best to instruct each group separately. Starting with buyers, assign each one a certain amount of money on their recording sheet, an example is provided in Figure 1. Tell them that they will be given a certain amount of money at the beginning of each round. Emphasize that the money can only be spent during one round. Any money not spent cannot be carried over to another round. This makes it clear that there is no benefit to holding money from round to round. Again, the instructor should emphasize that

[4] Many textbooks use Y to represent GDP or output. In this game, output produced is the same as output sold. This implies that changes in inventories do not need to be addressed.

[5] We thank both referees for pointing out that fake currency could also be used in place of cards with endowments. Potential benefits of this method are reducing recording errors and forcing students to not use fractional units of money.

[6] The banning of reselling and rebuying is not crucial, but it does minimize the chances of confusion.

the buyer's objective is to try to purchase as many units of goods from the sellers. Their overall score for the game will be the number of goods they are able to purchase across all rounds of the game.

Figure 1: Sample Buyer Sheet

BUYER – NAME:

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9	Round 10
	Money:	Money:	Money:	Money:	Money:	Money:	Money:	Money:	Money:	Money:
1										
2										
3										
4										
5										
6										
7										
8										
	Money Left:	Money Left:	Money Left:	Money Left:	Money Left:	Money Left:	Money Left:	Money Left:	Money Left:	Money Left:

Buyers should write down the price of each good they purchase. This information is crucial for the analysis of the game but also helps students account for their money. Although not necessary for the game, we also find it helpful for the students to write the amount of money they have left over. Once again, this encourages them to spend all their money. In some games, at least one student will spend more money than they are given. To minimize the probability of this mistake, the instructor can warn students that if they spend more money than they are given, they will forfeit the goods they purchased from that round.

The sellers are then informed that they will be given a certain number of goods to be sold in each round. This is specified at the top of the column of each round they play; see Figure 2. It is important to emphasize that any good not sold cannot be carried over to subsequent rounds. Again, this implies that sellers should try to sell all their goods in each round. After a student sells a good, the price should be written on their recording sheet. At the end of the round, each student records the number of goods not sold. As with the buyers, the instructor can tell the students they will forfeit the money they earned from that round if they sell more goods than they are allocated. The seller's overall score for the game will be the total amount of money they earn across all rounds of the game. Therefore, their objective is to maximize the amount of money they earn through selling their allocation of goods.

Figure 2: Sample Seller Sheet

SELLER – NAME:

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	Round 8	Round 9	Round 10
	Goods:	Goods:	Goods:	Goods:	Goods:	Goods:	Goods:	Goods:	Goods:	Goods:
1										
2										
3										
4										
5										
6										
7										
8										
	Goods Left:	Goods Left:	Goods Left:	Goods Left:	Goods Left:	Goods Left:	Goods Left:	Goods Left:	Goods Left:	Goods Left:

The money and good endowments are allocated to the buyers and sellers randomly after each round while keeping the roles the same. One method to do this is to have the cards shuffled and then given to students. The key considerations in choosing the values for the money and good allocations is that students should be able to buy and sell multiple times in a round and that the average price should be a value that facilitates transaction. If students are only able to buy or sell once or possibly zero times per round they will not be as engaged in the game. A simple way to accomplish this is to assign only two possible amounts of good and money allocations for the sellers and buyers respectively and to have an equal number of buyers and sellers. Two possible values simplify the required accounting for the instructor, while still providing some variety to the students to keep them engaged in the game. For example, suppose half the sellers are assigned 3 goods and half 4 goods. From experience, typically the sellers sell all the goods they are allocated. This means that the average output per student will be 3.5. On the other hand, buyers typically only spend about 80% of the money they are allocated.

To obtain whatever average price the instructor desires, a specific stock of money must be created. To compute the appropriate money stock, we use a per capita variation of the quantity equation, insert the desired average price, and compute M. If the desired average price is \$4, then the money stock should be 17.5. This is computed as follows:

$$M = \frac{PY}{V} = \frac{(4)(3.5)}{(0.8)} = 17.5$$

Thus, this stock could be obtained by assigning half the buyers \$16 and half \$19.

Again, giving the students two possible values of money and goods seems to help them be engaged in the game, as the uncertainty behind what allocation they will receive causes them to anticipate what value they will receive from the shuffled pile of cards. The students are not told anything regarding the values of the cards given out to any of the other players. The importance of this will be discussed in the next section of the paper where the post-game classroom discussion is described.

After the students select their cards giving them their allocations for the round, they are instructed to write down the quantity of goods or money on the top of the column corresponding to the round on their worksheet. The market is then opened and the students are free to buy and sell from each other. It is helpful to tell the students that if they have spent or sold all that they have been allocated to move to the side of the class. This makes it easier for the remaining students who are trying to make transactions to find other students. As such, it is easy to see when the market is about to clear. Most of the time there are some students who are holding out for a better than average transaction. To encourage the market to clear, the instructor can tell the class that there is one minute left. This encourages students to spend and sell their remaining stock.

Once the round is over, the data is collected from all the transactions made in the round. While the instructor could collect the data from either the buyers or sellers, we prefer to gather the data from the buyers. If the sellers are asked about the prices for which they sold their goods, it would be possible to tell the value of their allotment of goods. The buyers, on the other hand, will have more variable quantities of goods purchased. This will help prevent other sellers from figuring out the value of the seller allocation cards.

The instructor should have a spreadsheet application opened to record the value of the prices. It is helpful, although not strictly necessary, to have the spreadsheet displayed on a projector. That way, when the instructor is recording the prices, the students can and make sure the correct prices are recorded. Figure 3 depicts what the spreadsheet looks like with the recorded data. Students who are buyers simply report the prices of all the goods they purchased. For example, they could say, "Five, three, four," if they purchased three goods for the price of five, three, and four. Recording the price of each good purchased in the rows of a column in the spreadsheet also allows you to calculate the number of goods sold by counting the rows in each column. Thus, the common knowledge among game players after each round is how many goods each buyer purchased and the prices of each of those goods.

Typically, ten rounds are played. The first five rounds are played with one set of money allocation cards. The second five are played with a second set of money allocation cards. The money stock is increased by changing the allocations on the cards by increasing them by a value around 50%. So, if the value on the money allocation cards in the first five rounds were 16 and 19, this could be increased to 24 and 27. While not necessary, we usually try to change the cards without the students noticing. This will help them see the effect of a surprise increase in the money supply. To take their attention off the cards, students are asked to calculate their score up to that point in the game. For the buyers, they simply count the number of goods they have purchased, and for the sellers the count how much money they have earned. While they are doing these calculations, the old set of money allocation cards are exchanged for the new set with higher values.

Figure 3: Sample Spreadsheet

	Round 1	Round 2	Round 9	Round 10
1	4	2	5	6
2	5	3	4	5
3	4	3	5	5
4	3	5	6	6
5	2	5	6	6
6	3	4	4	5
7	3	5	4	5
8	4	2	6	5
19	5	3	5	5
20	5	4	5	6
21	3	2	5	6
22	4		5	5
23				
Y	22	21	22	22
P	3.75	3.45	5.00	5.42
M	100	100	150	150
V	0.83	0.73	0.73	0.79

To make the game proceed more smoothly, the following suggestions may help. First, restrict students from buying and selling repeatedly from the same person. This can sometimes encourage students to buy multiple units for a fixed price. For example, they could buy 3 goods for 5 units of money. This can cause confusion when they calculate the per-unit price, resulting in mistakes. Second, require exchange prices to be in whole units of money. When students sell in fractions of units of money, they can be confused about how much money they have left, which can sometimes lead to them spending more money than they have been allocated for the round. Having them buy and sell from one student at a time in whole units of money prevents errors and confusion. It also has the added benefit of disseminating information about the prevailing price that emerges from the market. When students are calling out a single price it is easier to find another student to buy or sell from.

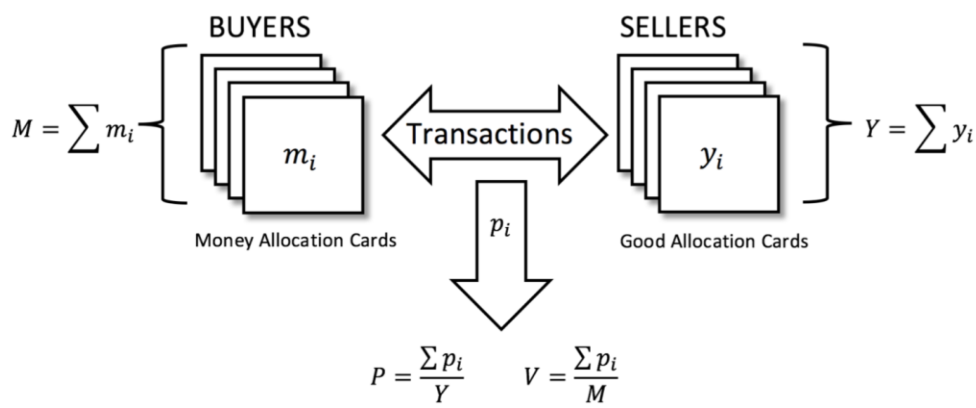
Game Analysis

After the tenth (or final) round, once the data on each round has been recorded in a spreadsheet, the results can be displayed for the students. First, the instructor can make a row below the recorded prices and label it Y. The students are told that this represents the number of goods produced in the economy, and that it can be calculated by counting the number of prices in each column. Next the instructor can calculate the average price for each round, or P. This can be displayed below the row with the values of Y. Figure 4 is a visual depiction of how values of M, V, P, and Y are calculated from the transactions in the game.

After calculating all the components of the quantity equation that come from the data collected, the instructor asks students if they can see any pattern or change in the average prices from the first round to the last. A 50% change in the money supply will almost certainly cause an increase in the average price level. After the increase has been noted by the students, the instructor can ask what caused that change. Most students will have noticed that the money allotment cards after the fifth round had higher values. The instructor can then reveal that the money supply was increased after the fifth round, and that it was the cause of the inflation. The money stock can be displayed below the row with the values of the average price level.

The instructor can then point out that this data can be applied to the quantity equation of money. After reminding the students of the formula and the definitions of each variable, the instructor can point out that three of the four variables in the equation have been calculated, M, P, and Y. The last variable to be calculated is velocity or V. Because velocity can be challenging for undergraduate students to comprehend, it can be helpful to remind them of the definition of velocity. We have found it instructive to ask the students if it is possible that the value of velocity in the game was greater than one. Usually, a student will answer that it is not possible, because the money could only be used once. We then ask the students if they think the value of velocity in the game was one. Again, usually a student will point out that buyers often had money left over and it wasn't used, so velocity must be less than one.

Figure 4: Game Schematic



To calculate velocity, simply take the ratio of the money spent to the total value of the money supply. To calculate the money spent, just add up all the prices reported in each round. Once this is done, velocity can be reported below the row that contains the information about M. The instructor at this point can ask the students if they see any interesting changes in the values of velocity across the rounds. Usually, the sixth round has the lowest value of velocity because students were accustomed to low prices and then suddenly were given more money. They purchased the same number of goods at the prices they were used to even though more money was introduced into the market. This unspent money lowers the velocity for that round. The values of velocity usually are similar between the fifth and the tenth round, as students start spending almost all their money to maximize the number of goods they can purchase.

Using the phrase "quantity equation" could be misleading. It is an identity which is a specific type of equation that is true by definition. Most equations that economists use are based upon an equilibrium condition, which is not true for the quantity equation. Thus, velocity is defined as the ratio of PY to M.[7] To drive this point home to the students, the instructor can solve for P noting that

$$P = \frac{MV}{Y}.$$

Then the following question can be presented to the students, "If the equation were to be tested and the values of M, V, and Y from the game were to be used in the equation above to estimate the actual value of P that the resulted from the game, how close would the value of MV/Y be to the actual value of P?" The two values will always be identical. In practice this tends to surprise students. It is impactful that the outcome of a set of uncoordinated interactions between students could be perfectly described by a simple equation. The instructor can then point out that the result will always hold. It can be pointed out that, when going over the results of the game, V was calculated using the following equation,

$$V = \frac{PY}{M}.$$

The "estimate" of P was calculated as

$$P = \frac{MV}{Y}.$$

[7] We thank a referee for pointing out that the ratio of MV to Y is actually income velocity, or the average number of times a unit of money is spent on final goods and services. This is compared to transaction velocity, which also includes expenditures of previously produced goods.

As was discussed above, P can also be calculated as the average of all prices resulting from the game. Next, substituting the value of V from above yields

$$P = \frac{MV}{Y} = M\left(\frac{PY}{M}\right)/Y = P.$$

Thus, the equation is true by definition.

The instructor should also emphasize to the students that this "trick" should not undermine the strong predictive power of the quantity equation. The increase in the money supply did cause an increase in the price level. This is one of the central messages of the quantity equation. After playing the game, it should be clearer to the students that inflation is "too much money, chasing too few goods." Understanding how an increase in the money supply causes a rise in overall prices is one of the key learning outcomes of this game. In addition, the results of this game can be used to illustrate the long run neutrality of money. Changes in the money supply were not related to Y in any way.

Further Analysis for Classroom Discussion

Once the basic framework of the quantity equation is understood by students, numerous issues can be brought up for discussion. For example, suppose the number of goods were increased in the second half of rounds while keeping the money stock constant; what would be the impact? The students can now apply the intuition behind the increase in the money supply causing an increase in the price level to this question. In the sixth round, buyers expected the same price level, but in aggregate had more money. This left them with extra money. In subsequent rounds, they spend that extra money on the same number of goods, thus increasing the price. If the number of goods were increased instead, then the same chain of reasoning could be followed. Sellers would then be left with extra goods when selling at the price level from the first five rounds. To get rid of those goods, they would then be willing to accept a lower price. Thus, an increase in goods would tend to lower the price level. This can be confirmed by using the quantity equation after solving for P and noting that Y is in the denominator. Thus, if the money supply and output rise at the same rate, the price level should remain constant. This, of course, assumes that velocity is constant.

As mentioned, another strength of this game is helping students acquire a deeper understanding of the velocity concept. In this game, after the post-game discussion, it is clear to students that the limit on velocity is one. Thus, velocity can be calculated without using the other three variables, given that velocity is just the average number of times a unit of money is used in a transaction. This helps the students articulate what velocity is without referring to the quantity equation. It is also a small step from there for them to understand the impact on velocity if students could move money across the rounds. However, more importantly, it is highly intuitive that if the students played this game a hundred or a thousand times, with exactly the same variables, then velocity would evolve to a constant. Once they understand that, then Friedman's quotation becomes comprehensible, as does the basic policy of *monetarism*. For inflation to persist, then the money stock must be rising faster than the stock of goods.

Once the impact of a constant velocity is established, the question can be raised of what happens when velocity suddenly changes, and why would velocity suddenly change? At this stage it is important to stress that the right-hand side of the money equation is equivalent to nominal GDP. In 2008 there was a 5% decline in nominal GDP, implying that either the money supply or velocity (or both) must have fallen. Given that the data indicates the money supply did not fall, then it must have been the case that velocity did, and thus the reason nominal GDP fell so dramatically was that the Federal Reserve did not increase the money supply to offset the decline in velocity. With this analysis the students are introduced to *market monetarism* and the idea of nominal GDP targeting. This game provides an ideal framework for helping students understand recent debates on economic events.

Conclusion

In this paper, we have introduced a simple classroom game that developed a deep of understanding the quantity equation, especially the concept of the velocity of money. As a result, students understand one theory of the causes of inflation as well as public policies such as monetarism and market monetarism. Understanding the causes of inflation should be a core learning objective in any introductory macroeconomics course. Such causes are not obvious, and also not well discussed in the media. As such, this exercise provides an excellent opportunity to display the benefits of learning economics and the powerful role of theory.

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A VBA Solution to Modern Portfolio Theory

Eric Girard and Rick Proctor[1]

ABSTRACT

The purpose of this article is to provide finance instructors an example of how to teach students to integrate Visual Basic for Application (VBA) into a modern portfolio theory application. Using tactical asset allocation as an example, we show how to teach students to use Excel to (1) collect refreshable data, (2) organize the data into input, (3) construct an efficient frontier, and (4) use VBA to automate the process. Our step-by-step methodology is intuitive and can be used for teaching how to integrate VBA into other dynamic and integrative financial models.

Introduction

Our paper is a tutorial. Its purpose is to address a complaint echoed by many employers in the finance industry. While graduating students may have the knowledge base to enter the workforce, they often lack the technical computing and data manipulation skills required to make an immediate contribution to their firm. For example, new finance graduates with basic Excel skills may only qualify for the most basic entry-level positions, while those with sophisticated computing skills such as VBA programming can start at more advanced positions where they are an immediate asset to the corporation.

Advanced Excel skills provide students with a competitive edge. Authors such as Girard and Ferreira (2011), Benninga (2008), Bauer (2006), MacDougall and Follows (2006), Whitworth (2010), Matsumoto et al. (2014) develop guidelines for integrating Excel applications to help smooth the grasp of finance theories. They all advocate that financial modeling not only bridges the gap between theories and concrete understanding, but also helps develop logical and analytical thinking skills, and the ability to synthesize.

Our tutorial allows the students to apply modern portfolio theory learned in the classroom to real-world, real-time data, helping to develop their advanced Excel skills while making the concept more concrete and understandable. We provide a step-by-step guide for instructors to teach students with little to no programming skills how to build a dynamic allocation model designed to implement a tactical sector allocation strategy. We show how to substantially reduce or eliminate the problems that students may have in combining the steps of downloading and manipulating the data from the internet to construct the efficient frontier and the capital market line (CML). This step-by-step approach is relatively straightforward to apply and streamlines a rather complex underlying theory. It has been very popular among the students, and it is worthwhile passing on to others who also teach investments.

The paper is organized as follows: In the next section, we discuss the tutorial's learning goals and outcomes; the third section shows how to compute and calibrate the risk and return for a portfolio of numerous assets; in the fourth section, students are taught how to perform a constrained optimization in Excel, and how to use VBA to automate the building of the efficient frontier and the CML. The fifth section concludes.

Goals and Measurable Learning Outcomes

When teaching modern portfolio theory, we often find our students struggling to conceptualize the application of the efficient frontier, portfolio utility, and the CML. In addition, textbooks do not show the process of tranforming financial data into an efficient frontier and a CML.[2] In this paper, we share our

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[2] An excellent textbook on financial modeling is Benninga (2008). However, those familiar with that textbook will find that our step-by-step guidelines are easier to follow and more cohesive since they combine all of the different steps necessary to the process.

experience in bridging the gap between a complex theory and its application. Our students have access to this tutorial and its accompanying video.[3] Our role as instructors consists of introducing them to this learning experience, and making sure that they can replicate it in a different context.

Following a lecture, students are shown how to perform a constrained optimization during a laboratory session. Using the example described in this paper, we show students how to (1) compute expected returns, risks, and correlations for each asset class, and (2) use these metrics to determine the optimal asset mix—i.e., recommended weight for each asset class.

Then, students are given a scenario – based on a simplified hedge fund disclosure statement or a hypothetical policy statement – providing information about asset classes' returns, risks, and co-movements, which are input into an optimization process to generate asset class weights. For instance, the tactical U.S. sector allocation scenario used in this paper is adapted from a disclosure statement based on global tactical asset allocation. The reason behind tactical asset allocation is to address the importance of (1) the timing of investment decisions, and (2) the integration of the process for retrieving and transforming financial data, feeding it into an optimization procedure, and visualizing the output.[4]

Finally, they are asked to implement the process described in the scenario using real data retrieved from Capital IQ®. Of course, the type and number of asset classes in this assignment are different from the ones used in the tutorial. They submit their model and a short narrative explaining what they did, what they found, and the implications of their findings. Students are graded on the model's execution and the narrative's content.

Upon completion of this assignment, students have created an integrated model to (1) compute and calibrate the risk and return for a portfolio of numerous assets, (2) find the optimal asset mix, and (3) build the efficient frontier and the CML. The benefits of this exercise are two-fold: First, students' comprehension of the theory is enhanced; second, they develop a set of computer programming skills that can be utilized in other modeling applications. Our school's finance curriculum follows the Chartered Financial Analyst (CFA) body of knowledge; accordingly, this assignment's learning outcomes are 2018 CFA level 1 LOS 41 "c through h," LOS 42 "a and b," LOS 43 "a through c," and LOS 43 "f."[5]

Organizing the Process: Input and Output

The first phase of the process consists of retrieving the data needed to compute the inputs for constructing an optimal asset mix. The assets used in our example are the stocks populating the ten S&P 500 sector indices. Using data from Capital IQ®,[6] we create a dynamic data feed and show how to compute expected returns, standard deviations, and pairwise correlations for the ten S&P500 economic sectors: Consumer Discretionary, Consumer Staples, Energy, Financials, Healthcare, Industrials, Information Technology, Materials, Telecommunications, and Utilities.

This first phase consists of two steps. First, we download historical time series on sector indices, transforming these price series into return series, and calculating pairwise covariances. We are assuming that future returns, variance, and covariance revert to their long-term averages. Second, we compute the 1-year target return for each stock populating the S&P500 using analysts' mean target price and dividends, and aggregate these short-run expectations by economic sector; we use these 1-year estimates to capture relative short-run mispricing.

Before we start building the model, some preliminary actions need to be taken. In a new "macro-enabled" workbook, we create three worksheets named "Historical Sector Data," "Forecast," and "Optimization."

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[3] They have access to (1) a shorter version of this paper that only states all the necessary Excel operations, and (2) a video created with a screen capture software showing how to implement the example used in the tutorial.

[4] Ibbotson (2010) and Xiong et al. (2010) demonstrated that the two most important elements affecting future portfolio performance are (1) the composition and (2) the timing of the asset mix. These are the premises underlying tactical asset allocation strategies—i.e., to provide investors with an opportunity to benefit from relative misvaluations that exist between and within capital markets.

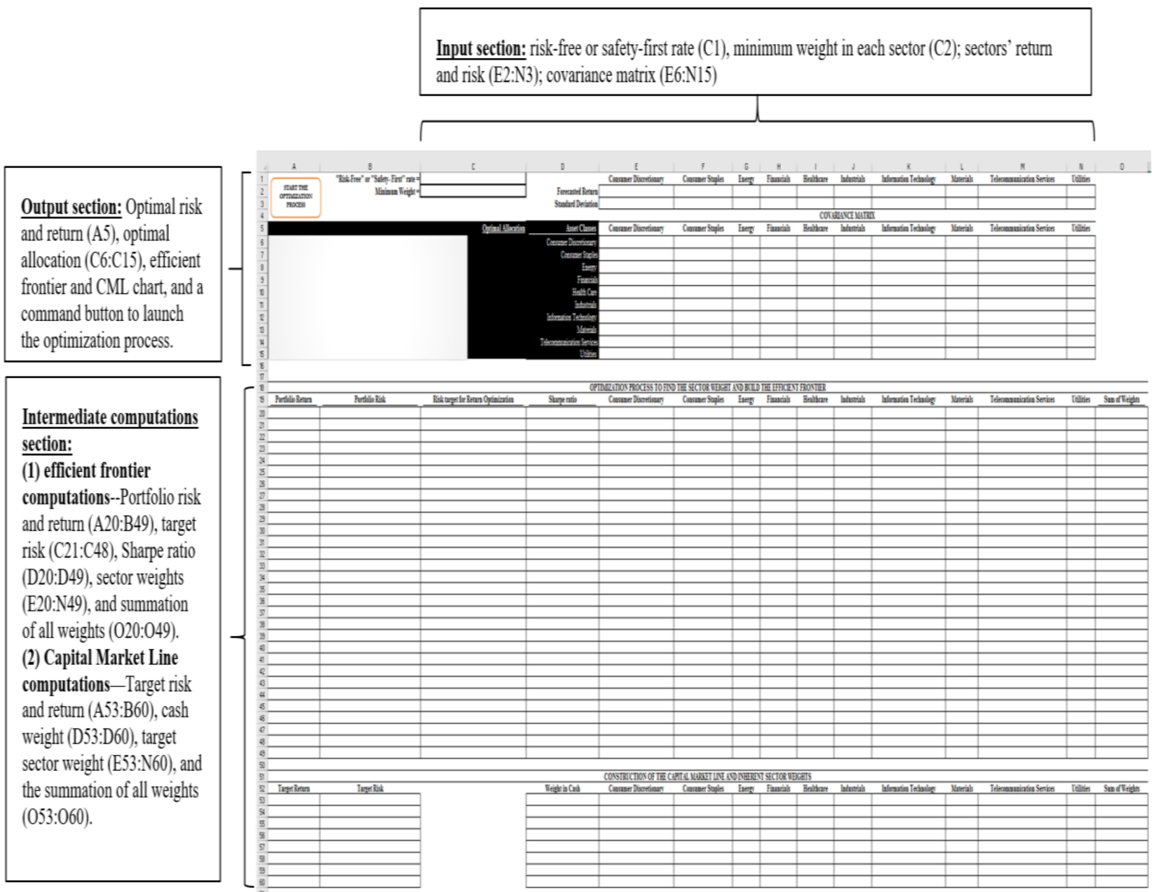
[5] The learning outcomes are available at https://www.cfainstitute.org/CFA%20Program%20Study%20Session/2018_L1_SS12.pdf.

[6] Other commercial data feeds such as Bloomberg, Factset, Compustat, or CRSP have their own Excel interface. Free historical data, such as Yahoo Finance or MSCI data, are also available, but the choice and quality of time series is limited. VBA routines for automatic download from free sites are readily available; e.g., the "code project" provides several useful routines to systematize the downloading of Yahoo.finance data (http://www.codeproject.com/Articles/740069/VBA-Macros-Provide-Yahoo-Stock-Quote-Downloads-in).

Then, all necessary Excel functions and tools are set up—i.e., the Developer ribbon is made visible, the Solver add-in is loaded, and Solver VBA references are made available.[7]

While input in worksheets "Historical Sector Data" and "Forecast" are idiosyncratic to the data source available, the worksheet "Optimal Allocation" is the same regardless of the input. The building of this worksheet is essential to the exercise, and its mapping is shown in Figure 1. Inputs from the two other worksheets are aggregated in range E2:N15; the risk-free rate (or safety-first) assumption and short sale constraint are in range C1:C2; intermediate computations necessary to build the efficient frontier and the CML are in the range A20:O60, and the output is summarized in range A5:C15.

Figure 1: Optimization Worksheet Blueprint



If instructors prefer to make the process more concise and less overwhelming, they can provide students with the asset classes' return, risk, and covariance data in range E2:N15. We include the building steps of the "Historical Sector Data" worksheet, "Forecast" worksheets, and the process for computing each sector's return, risk, and pairwise covariance in the appendix. The rest of the paper focuses on the "Optimization" worksheet.

Portfolio Risk and Return

At this point, we recommend reminding students of (1) the computation of a portfolio return and standard deviation, and (2) the definition of the market portfolio, and (3) the mathematics of the CML.

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[7] To make the **Developer** ribbon visible, select **File** in the menu bar, **Options**, **Customized Ribbon**, and check the box next to **Developer** on the right-end side. To Load the Solver add-in, select **File** in the menu bar, **Options**, **Add-in**, press the **GO** button at the bottom of the window, check the box next to **Solver add-in**, and press the **OK** button. To ensure that the VBA editor has access to Solver references, click the **Visual Basic** button on the **Developer** ribbon, select **Tools** in the visual basic editor menu bar, press **References** in the drop-down menu, check the box next to **Solver**, and press the **OK** button.

Let ω_i be the proportion (weight) invested in asset i, r_i the expected return on asset i, σ_{ij} the covariance between the returns of asset i and asset j, and G is minimum allowable weight amount. That is, with n as the number of securities in a portfolio, G∊[-∞;1/n]—i.e., when G<0, short sales are allowed; otherwise, all investments consist of long positions.

Let Γ^T be the transpose of the weight vector ($[\omega_1 \ldots \omega_n]$), R, the return vector ($[r_1 \ldots r_n]$), and S, the covariance matrix ($\begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \cdots & \cdots & \cdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{bmatrix}$). A portfolio return ($R_p$), its standard deviation (σ_p), and its reward-to-risk (RTR) have the following formulas:

$$R_p = \Gamma R,$$
$$\sigma_p = (\Gamma^T S \Gamma)^{½}$$
$$RTR = (\Gamma R - C)/(\Gamma^T S \Gamma)^{1/2}$$

where RTR is the Sharpe ratio when C is the risk-free rate; it is the Roy (1952) safety first ratio when C is a minimum required rate of return (safety-first criteria).

Modern portfolio theory states that the market portfolio is the only point of interest when building an efficient frontier. The market portfolio is the portfolio with the maximum reward-to-risk; its return is R_m, and its risk is σ_m.

The CML is created using a mix between the risk-free asset (cash) and the market portfolio.[8] Accordingly, a specific market portfolio and cash mix defines each targeted risk level and the cash weight (ω_c) for each targeted risk is computed as

$$\omega_c = 1 - \sigma_P/\sigma_M$$

We recommend highlighting a few practical caveats related to MPT's implementation. First, when (1) the risk-free rate is replaced by a minimum required rate of return or (2) the optimal portfolio does not consider the universe's asset mix, the CML is a Capital Allocation Line (CAL). Second, based on utility theory, an investor's coefficient of risk aversion defines his/her risk tolerance. Further, the risk aversion coefficient in the (risk) utility function is negatively correlated with the range of risk targets used in this exercise.

Figure 2 shows the formulas for the portfolio return, risk, target risk,[9] and Sharpe ratio in range A20:D49. The summation of all weights is computed in range O20:O49. Row 20 is reserved for the portfolio with the smallest risk (global minimum risk portfolio) and row 49 for the portfolio with the highest Sharpe ratio (the market portfolio). Computations are as follows:

- To calculate the portfolio return, type **=SUMPRODUCT(E$2:N$2,E20:N20)** in cell A20; copy cell A20 and paste it into range A21:A49.

- To compute the portfolio standard deviation, type **=(12*MMULT(MMULT(E20:P20,$E$6:$N$15),TRANSPOSE(E20:P20)))^0.5** in cell B20, then press concurrently the keys **CTRL-SHIFT-ENTER**; copy B20, and paste it into range B21:B49.

- The targeted risk values increase by equal increments over 28 rows. They span from the smallest standard deviation in cell B20 to the standard deviation of the sector with the highest return in range E2:N3. To calculate the values in range C21:C48, type **= B20+(HLOOKUP(MAX(E$2:N$2),E$2:N$3,2,FALSE)-B$20)/28** in cell C21; copy cell C21, and paste it into range C22:C48.

- To compute the Sharpe ratio, type **=(A20-$B$1)/B20** in cell D20; copy cell D20 and paste it into range D21:D49.

- Type **=SUM(E20:N20)** in cell O20; copy cell O20, and paste it into range O21:O49. The sectors' weights in range E20:N49 are empty. An optimization process will later populate these cells.

---

[8] Students can be reminded that a portfolio that combines the optimal portfolio M and the risk free asset has a standard deviation of $\sigma_p = (1 - \omega_{Rf}) \times \sigma_m$ and a return of $R_p = R_f + \sigma_p (Rm - R_f)/\sigma_m$, which is the equation for the CML.

[9] Range C21:C48 is used to constrain the efficient frontier's construction within an (economically) feasible range of risk values. Indeed, a portfolio's risk lies between the global minimum risk and the standard deviation of the asset class with the highest return.

Figure 2: Portfolio Return, Risk, Target Risk and Sharpe Ratio Formulas

	A	B	C	D	O
16					
17					
18	OPTIMIZATION PROCESS TO FIND THE SECTOR WEIGHT AND BUILD THE EFFICIENT FRONTIER				
19	Portfolio Return	Portfolio Risk	Risk target for Return Optimization	Sharpe ratio	Sum of Weights
20	=SUMPRODUCT($E$2:$N$2,E20:N20)	=(12*(MMULT(MMULT(E20:N20,$E$6:$N$15),TRANSPOSE(E20:N20))))^0.5		=(A20-$C$1)/B20	=SUM(E20:N20
21	=SUMPRODUCT($E$2:$N$2,E21:N21)	=(12*(MMULT(MMULT(E21:N21,$E$6:$N$15),TRANSPOSE(E21:N21))))^0.5	=B20+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A21-$C$1)/B21	=SUM(E21:N21
22	=SUMPRODUCT($E$2:$N$2,E22:N22)	=(12*(MMULT(MMULT(E22:N22,$E$6:$N$15),TRANSPOSE(E22:N22))))^0.5	=B21+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A22-$C$1)/B22	=SUM(E22:N22
23	=SUMPRODUCT($E$2:$N$2,E23:N23)	=(12*(MMULT(MMULT(E23:N23,$E$6:$N$15),TRANSPOSE(E23:N23))))^0.5	=B22+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A23-$C$1)/B23	=SUM(E23:N23
24	=SUMPRODUCT($E$2:$N$2,E24:N24)	=(12*(MMULT(MMULT(E24:N24,$E$6:$N$15),TRANSPOSE(E24:N24))))^0.5	=B23+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A24-$C$1)/B24	=SUM(E24:N24
25	=SUMPRODUCT($E$2:$N$2,E25:N25)	=(12*(MMULT(MMULT(E25:N25,$E$6:$N$15),TRANSPOSE(E25:N25))))^0.5	=B24+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A25-$C$1)/B25	=SUM(E25:N25
26	=SUMPRODUCT($E$2:$N$2,E26:N26)	=(12*(MMULT(MMULT(E26:N26,$E$6:$N$15),TRANSPOSE(E26:N26))))^0.5	=B25+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A26-$C$1)/B26	=SUM(E26:N26
27	=SUMPRODUCT($E$2:$N$2,E27:N27)	=(12*(MMULT(MMULT(E27:N27,$E$6:$N$15),TRANSPOSE(E27:N27))))^0.5	=B26+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A27-$C$1)/B27	=SUM(E27:N27
28	=SUMPRODUCT($E$2:$N$2,E28:N28)	=(12*(MMULT(MMULT(E28:N28,$E$6:$N$15),TRANSPOSE(E28:N28))))^0.5	=B27+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A28-$C$1)/B28	=SUM(E28:N28
29	=SUMPRODUCT($E$2:$N$2,E29:N29)	=(12*(MMULT(MMULT(E29:N29,$E$6:$N$15),TRANSPOSE(E29:N29))))^0.5	=B28+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A29-$C$1)/B29	=SUM(E29:N29
30	=SUMPRODUCT($E$2:$N$2,E30:N30)	=(12*(MMULT(MMULT(E30:N30,$E$6:$N$15),TRANSPOSE(E30:N30))))^0.5	=B29+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A30-$C$1)/B30	=SUM(E30:N30
31	=SUMPRODUCT($E$2:$N$2,E31:N31)	=(12*(MMULT(MMULT(E31:N31,$E$6:$N$15),TRANSPOSE(E31:N31))))^0.5	=B30+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A31-$C$1)/B31	=SUM(E31:N31
32	=SUMPRODUCT($E$2:$N$2,E32:N32)	=(12*(MMULT(MMULT(E32:N32,$E$6:$N$15),TRANSPOSE(E32:N32))))^0.5	=B31+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A32-$C$1)/B32	=SUM(E32:N32
33	=SUMPRODUCT($E$2:$N$2,E33:N33)	=(12*(MMULT(MMULT(E33:N33,$E$6:$N$15),TRANSPOSE(E33:N33))))^0.5	=B32+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A33-$C$1)/B33	=SUM(E33:N33
34	=SUMPRODUCT($E$2:$N$2,E34:N34)	=(12*(MMULT(MMULT(E34:N34,$E$6:$N$15),TRANSPOSE(E34:N34))))^0.5	=B33+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A34-$C$1)/B34	=SUM(E34:N34
35	=SUMPRODUCT($E$2:$N$2,E35:N35)	=(12*(MMULT(MMULT(E35:N35,$E$6:$N$15),TRANSPOSE(E35:N35))))^0.5	=B34+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A35-$C$1)/B35	=SUM(E35:N35
36	=SUMPRODUCT($E$2:$N$2,E36:N36)	=(12*(MMULT(MMULT(E36:N36,$E$6:$N$15),TRANSPOSE(E36:N36))))^0.5	=B35+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A36-$C$1)/B36	=SUM(E36:N36
37	=SUMPRODUCT($E$2:$N$2,E37:N37)	=(12*(MMULT(MMULT(E37:N37,$E$6:$N$15),TRANSPOSE(E37:N37))))^0.5	=B36+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A37-$C$1)/B37	=SUM(E37:N37
38	=SUMPRODUCT($E$2:$N$2,E38:N38)	=(12*(MMULT(MMULT(E38:N38,$E$6:$N$15),TRANSPOSE(E38:N38))))^0.5	=B37+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A38-$C$1)/B38	=SUM(E38:N38
39	=SUMPRODUCT($E$2:$N$2,E39:N39)	=(12*(MMULT(MMULT(E39:N39,$E$6:$N$15),TRANSPOSE(E39:N39))))^0.5	=B38+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A39-$C$1)/B39	=SUM(E39:N39
40	=SUMPRODUCT($E$2:$N$2,E40:N40)	=(12*(MMULT(MMULT(E40:N40,$E$6:$N$15),TRANSPOSE(E40:N40))))^0.5	=B39+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A40-$C$1)/B40	=SUM(E40:N40
41	=SUMPRODUCT($E$2:$N$2,E41:N41)	=(12*(MMULT(MMULT(E41:N41,$E$6:$N$15),TRANSPOSE(E41:N41))))^0.5	=B40+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A41-$C$1)/B41	=SUM(E41:N41
42	=SUMPRODUCT($E$2:$N$2,E42:N42)	=(12*(MMULT(MMULT(E42:N42,$E$6:$N$15),TRANSPOSE(E42:N42))))^0.5	=B41+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A42-$C$1)/B42	=SUM(E42:N42
43	=SUMPRODUCT($E$2:$N$2,E43:N43)	=(12*(MMULT(MMULT(E43:N43,$E$6:$N$15),TRANSPOSE(E43:N43))))^0.5	=B42+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A43-$C$1)/B43	=SUM(E43:N43
44	=SUMPRODUCT($E$2:$N$2,E44:N44)	=(12*(MMULT(MMULT(E44:N44,$E$6:$N$15),TRANSPOSE(E44:N44))))^0.5	=B43+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A44-$C$1)/B44	=SUM(E44:N44
45	=SUMPRODUCT($E$2:$N$2,E45:N45)	=(12*(MMULT(MMULT(E45:N45,$E$6:$N$15),TRANSPOSE(E45:N45))))^0.5	=B44+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A45-$C$1)/B45	=SUM(E45:N45
46	=SUMPRODUCT($E$2:$N$2,E46:N46)	=(12*(MMULT(MMULT(E46:N46,$E$6:$N$15),TRANSPOSE(E46:N46))))^0.5	=B45+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A46-$C$1)/B46	=SUM(E46:N46
47	=SUMPRODUCT($E$2:$N$2,E47:N47)	=(12*(MMULT(MMULT(E47:N47,$E$6:$N$15),TRANSPOSE(E47:N47))))^0.5	=B46+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A47-$C$1)/B47	=SUM(E47:N47
48	=SUMPRODUCT($E$2:$N$2,E48:N48)	=(12*(MMULT(MMULT(E48:N48,$E$6:$N$15),TRANSPOSE(E48:N48))))^0.5	=B47+(HLOOKUP(MAX($E$2:$N$2),$E$2:$N$3,2,FALSE)-$B$20)/28	=(A48-$C$1)/B48	=SUM(E48:N48
49	=SUMPRODUCT($E$2:$N$2,E49:N49)	=(12*(MMULT(MMULT(E49:N49,$E$6:$N$15),TRANSPOSE(E49:N49))))^0.5		=(A49-$C$1)/B49	=SUM(E49:N49

To build the CML, we combine the information contained in range A49:N49 (the optimal portfolio) and cell C1 (risk-free asset or Safety-first criterion) to calculate portfolio returns given several target risk levels.

Figure 3: Formulas to Construct the Capital Market Line

	A	B	D	E	F	G	H	I	J	K	L	M	N	O
51	CONSTRUCTION OF THE CAPITAL MARKET LINE AND INHERENT SECTOR WEIGHTS													
52	Target Return	Target Risk	Weight in Cash	Consumer Discr	Consumer Stapl	Energy	Financials	Healthcare	Industrials	Information Tecl	Materials	Telecommunicati	Utilities	Sum of Weights
53	=D53*$C$1+(1-D53)*$A$49	0	=1-B53/$B$49	=(1-$D53)*E$49	=(1-$D53)*F$4	=(1-$D53)*G$4	=(1-$D53)*H$4	=(1-$D53)*I$49	=(1-$D53)*J$49	=(1-$D53)*K$49	=(1-$D53)*L$49	=(1-$D53)*M$49	=(1-$D53)*N$49	=SUM(D53:N53)
54	=D54*$C$1+(1-D54)*$A$49	=25%*B49	=1-B54/$B$49	=(1-$D54)*E$49	=(1-$D54)*F$4	=(1-$D54)*G$4	=(1-$D54)*H$4	=(1-$D54)*I$49	=(1-$D54)*J$49	=(1-$D54)*K$49	=(1-$D54)*L$49	=(1-$D54)*M$49	=(1-$D54)*N$49	=SUM(D54:N54)
55	=D55*$C$1+(1-D55)*$A$49	=0.5*B49	=1-B55/$B$49	=(1-$D55)*E$49	=(1-$D55)*F$4	=(1-$D55)*G$4	=(1-$D55)*H$4	=(1-$D55)*I$49	=(1-$D55)*J$49	=(1-$D55)*K$49	=(1-$D55)*L$49	=(1-$D55)*M$49	=(1-$D55)*N$49	=SUM(D55:N55)
56	=D56*$C$1+(1-D56)*$A$49	=0.75*B49	=1-B56/$B$49	=(1-$D56)*E$49	=(1-$D56)*F$4	=(1-$D56)*G$4	=(1-$D56)*H$4	=(1-$D56)*I$49	=(1-$D56)*J$49	=(1-$D56)*K$49	=(1-$D56)*L$49	=(1-$D56)*M$49	=(1-$D56)*N$49	=SUM(D56:N56)
57	=D57*$C$1+(1-D57)*$A$49	=$B$49	=1-B57/$B$49	=(1-$D57)*E$49	=(1-$D57)*F$4	=(1-$D57)*G$4	=(1-$D57)*H$4	=(1-$D57)*I$49	=(1-$D57)*J$49	=(1-$D57)*K$49	=(1-$D57)*L$49	=(1-$D57)*M$49	=(1-$D57)*N$49	=SUM(D57:N57)
58	=D58*$C$1+(1-D58)*$A$49	=1.25*$B$49	=1-B58/$B$49	=(1-$D58)*E$49	=(1-$D58)*F$4	=(1-$D58)*G$4	=(1-$D58)*H$4	=(1-$D58)*I$49	=(1-$D58)*J$49	=(1-$D58)*K$49	=(1-$D58)*L$49	=(1-$D58)*M$49	=(1-$D58)*N$49	=SUM(D58:N58)
59	=D59*$C$1+(1-D59)*$A$49	=1.5*$B$49	=1-B59/$B$49	=(1-$D59)*E$49	=(1-$D59)*F$4	=(1-$D59)*G$4	=(1-$D59)*H$4	=(1-$D59)*I$49	=(1-$D59)*J$49	=(1-$D59)*K$49	=(1-$D59)*L$49	=(1-$D59)*M$49	=(1-$D59)*N$49	=SUM(D59:N59)
60	=D60*$C$1+(1-D60)*$A$49	=1.75*$B$49	=1-B60/$B$49	=(1-$D60)*E$49	=(1-$D60)*F$4	=(1-$D60)*G$4	=(1-$D60)*H$4	=(1-$D60)*I$49	=(1-$D60)*J$49	=(1-$D60)*K$49	=(1-$D60)*L$49	=(1-$D60)*M$49	=(1-$D60)*N$49	=SUM(D60:N60)

Figure 3 shows the formulas for target risk and return, and inherent allocation to cash and risky assets as follows:

- Choose a set of target risks in range B53:B60—i.e., 0%, 50%, 75%, 100%, 125%, 150%, and 175% of the market risk (cell B49);
- Compute the cash allocation for each level of risk by typing **=1-B53/B$49** in cell D53; copy cell D53, and paste it into range D54:D60.

• Calculate the allocation to the 10 U.S. sectors, by typing **=(1-$D53)*E$49** in cell E53; copy cell E53, and paste it into range E53:N60.

• Compute the portfolio weighted average return by typing **=D53*$B$1+(1-D53)*$A$49** in cell A53, copy cell A53, and paste it in range A54:A60.

Output Section

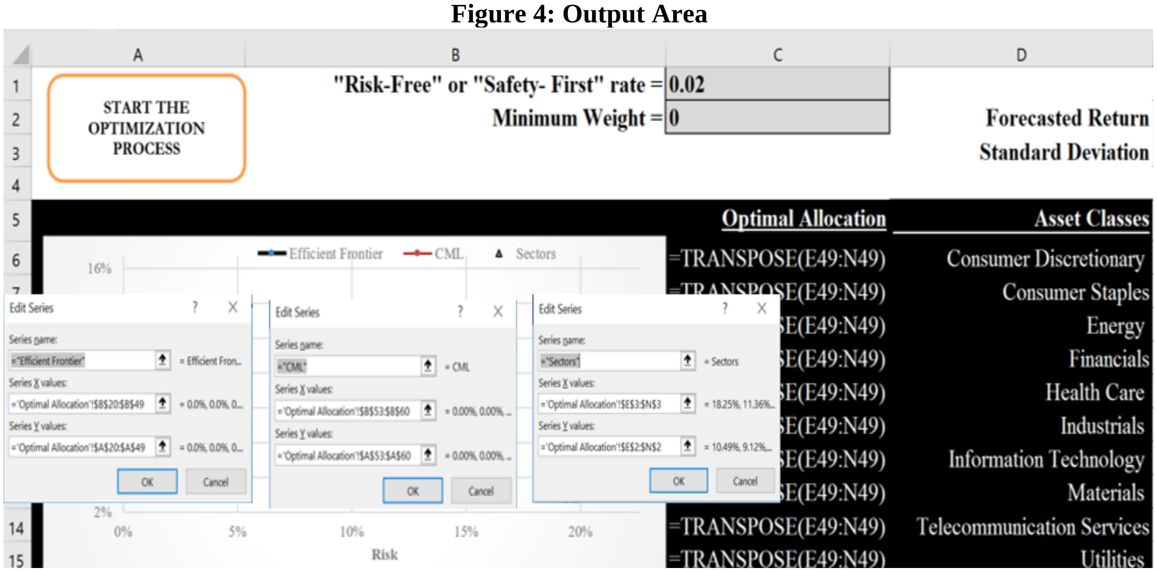
As shown in Figure 4, we place the output summary in range A5:C18. To highlight the coordinates of Portfolio M, type **="Optimal Portfolio: Return = "&ROUND(A49,2)&"%"&"; Risk = "&ROUND(B49,2)&"%"** in cell A5.

We insert the market portfolio composition in range D6:D15. Starting from cell D6, select range D6:D15, type **=TRANSPOSE(E49:N49)**, and press the three keys **CTRL-SHIFT-ENTER** concurrently.

For the graph, select a scatter plot, and add three series—i.e., the efficient frontier with B20:B49 on the x-axis and A20:A49 on the y-axis, the CML with B53:B60 on the x-axis and A53:A60 on the y-axis, and each risky asset with E3:N3 on the x-axis and E2:N2 on the y-axis.

Finally, we insert a shape in the top left corner above the chart. We will use it later to launch a macro.

Figure 4: Output Area



Optimization and Automation

In a nutshell, the construction of an efficient frontier consists of (1) maximizing a portfolio return for a given level risk by changing the weights of each asset class included in that portfolio, (2) repeating the process for many levels of portfolio risk, and (3) plotting each optimal risk-return combination.

Formally, an efficient frontier consists of a plot of all maximum portfolio returns for different levels of portfolio risk ranging from the global minimum risk (σ_{min}, hereafter) to the standard deviation of the asset with the highest return (σ_{max}, thereafter). The process of building an efficient frontier can be broken down into the following three operations:

1. Operation 1: Find the asset mix associated with σ_{min}. That is, minimize a portfolio standard deviation $(\Gamma^T S\Gamma)^{½}$ by changing the assets weight (Γ^T) under the following two constraints:

    • All weights are greater or equal to G ($\Gamma^T \geq G$), where G is the minimum allowable weight amount. If n is the number of securities in a portfolio, Gϵ[-∞;1/n]—i.e., when G<0, short sales are allowed; otherwise, all investments consist of long positions.

- All weights sum up to 1 ($\sum_{i=1}^{n} \omega_i$=1).

2. Operation 2: Determine the asset mix inherent to the maximum portfolio return given a targeted portfolio risk (σ_{target}). Then, continue the process for all possible levels of σ_{target} incrementally greater than σ_{min}, yet smaller than σ_{max}. That is, maximize a portfolio return (ΓR) by changing the assets weight (Γ^T) under the following three constraints:

- All weights are greater or equal to G ($\Gamma^T \geq G$),
- All weights sum up to 1 ($\sum_{i=1}^{n} \omega_i$=1), and
- The portfolio standard deviation $(\Gamma^T S\Gamma)^{½} = \sigma_{target}$, where $\sigma_{target} \in [\sigma_{min}; \sigma_{max}]$.

3. Operation 3: Find the asset mix associated with the market portfolio. That is, maximize a portfolio's RTR--$(\Gamma R\text{-}C)/(\Gamma^T S\Gamma)^{1/2}$-- by changing the assets weight (Γ^T) subjected to the following two constraints:

- All weights are greater or equal to G ($\Gamma^T \geq G$),
- All weights sum up to 1 ($\sum_{i=1}^{n} \omega_i$=1).

Next, we show how to implement these three operations. We start by recording our work by pressing the **Record Macro** button located in the **Developer** ribbon. Once the workbook is in the **Record Macro** mode, Excel records every single action in Visual Basic for Application (VBA) code. This code can be accessed, edited, modified, and re-arranged into a program that performs the entire process at the push of a button.

We recommend that students practice the next few steps before switching to **Record Macro** mode. We also remind them to confine themselves to the required operations to minimize the amount of code editing.

Optimization

We implement the three above-mentioned operations as follows:

Operation 1

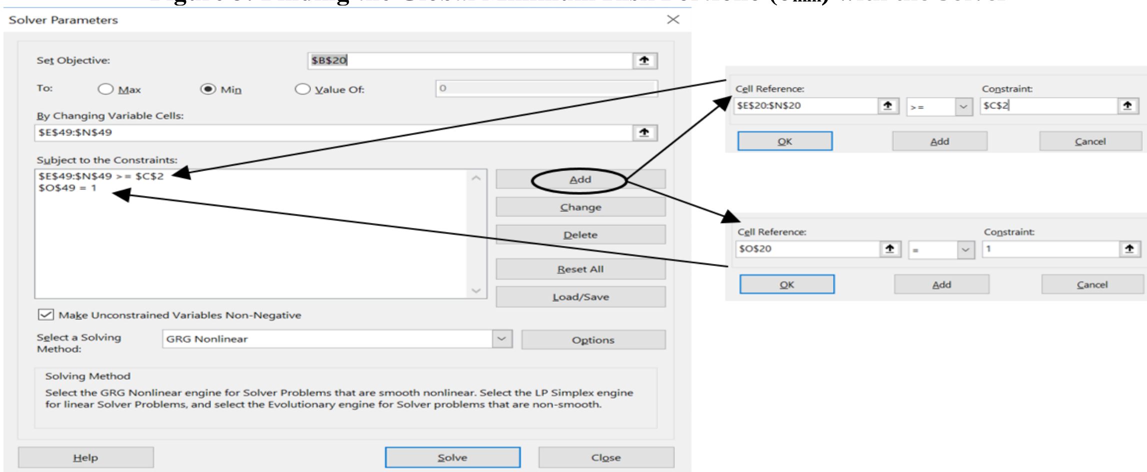
To find the global minimum variance portfolio, the optimization "objective" is to "minimize" a portfolio standard deviation by "changing" the weights of each asset class, "subject to the constraints:" (1) all weights add-up to 100 percent, and (2) weights are constrained for short-selling restrictions.

Row 20 is used to determine the global minimum risk portfolio. Select **Solver** under the **Data** ribbon. In the **Solver Parameters** dialog box, enter all input as shown in Figure 5:

- Press the **Reset All** button, then press the **OK** button (to erase previously stored input)
- Input cell B20 (portfolio standard deviation) in the **Set objective** box, then check the **Min** (minimum) button
- Input E20:N20 (sector weights) in the **By Changing Cells** box
- Under **Subject to the Constraints**:
  - Press the **Add** button. In the dialog box, select O20 as a **Cell Reference**, = in the middle box, type 1 in the **Constraint** box, and press the **OK** button (we are forcing the sum of all sector weights to add up to 100%).
  - Press the **Add** button. In the dialog box, select range E20:N20 as a **Cell Reference**, >= in the middle box, cell C2 as a **Constraint**, and press the **OK** button (sector weight restriction).
- Press the **Solve** button. The **Solver Results** dialog box appears. Choose **Keep Solver Solution** to

validate. The optimal weights appear in range E20:N20 and the related portfolio return, risk, and Sharpe ratio are automatically computed in range A20:D20.

Figure 5: Finding the Global Minimum Risk Portfolio (σ_{min}) with the Solver



Operation 2

To find a portfolio with the maximum return for a given level of risk, the optimization "objective" is to "maximize" the portfolio return by "changing" the weights of each asset class, "subject to the constraints:" (1) all weights add-up to 100 percent, (2) weights are constrained for short-selling restrictions, and (3) the portfolio standard deviation is equal to a given value.

Row 21 is used to determine the maximum portfolio return for a standard deviation slightly above the global minimum standard deviation.[10] Select **Solver** under the **Data** ribbon. In the **Solver Parameters** dialog box appears, enter all input as shown in Figure 6:

- Press the **Reset All** button, then press the **OK** button

- Input cell A21 in the **Set objective** box (portfolio return), and check the **Max** (maximum) button.

- Input E21:N21 (sector weights) in the **By Changing Cells** box

- Under **Subject to the Constraints**:

  - Press the **Add** button. In the dialog box, select O21 as a **Cell Reference**, = in the middle box, type 1 in the **Constraint** box, and press the **OK** button.

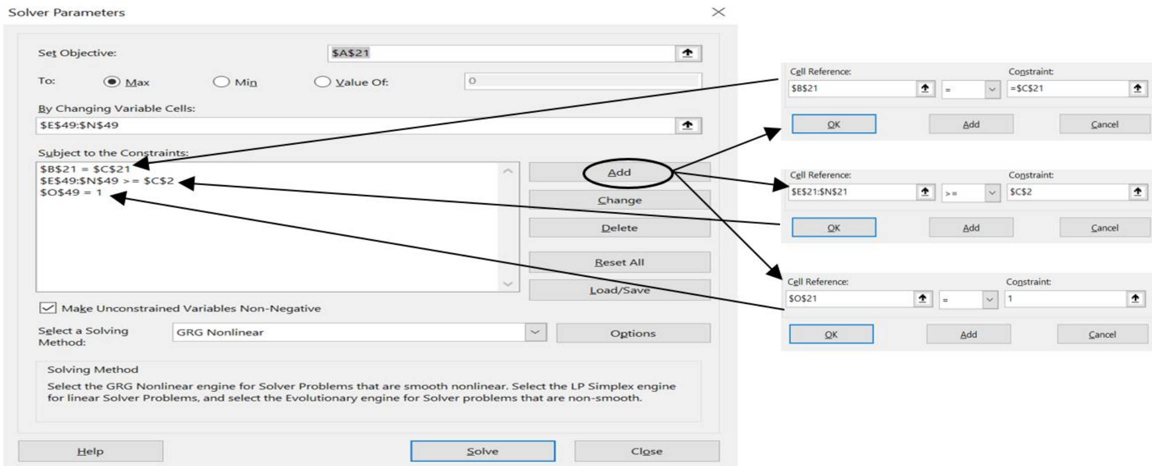
  - Press the **Add** button. In the dialog box, select range E21:N21 as a **Cell Reference**, >= in the middle box, cell C2 as a **Constraint**, and press the **OK** button.

  - Press the **Add** button. In the dialog box, select B21 as a **Cell Reference**, = in the middle box, input cell C21 in the **Constraint** box, and press the **OK** button (the standard deviation for which the portfolio return is maximized).

- Press the **Solve** button. The **Solver Results** dialog box appears. Choose **Keep Solver Solution** to validate. The optimal weights appear in range E21:N21 and the related portfolio return, risk, and Sharpe ratio are automatically computed in range A21:D21.

[10] A lengthy alternative to the proposed three-step process consists of repeating step 2 from row 22 to 48. We suggest showing students how to modify a VBA code to perform these computations automatically.

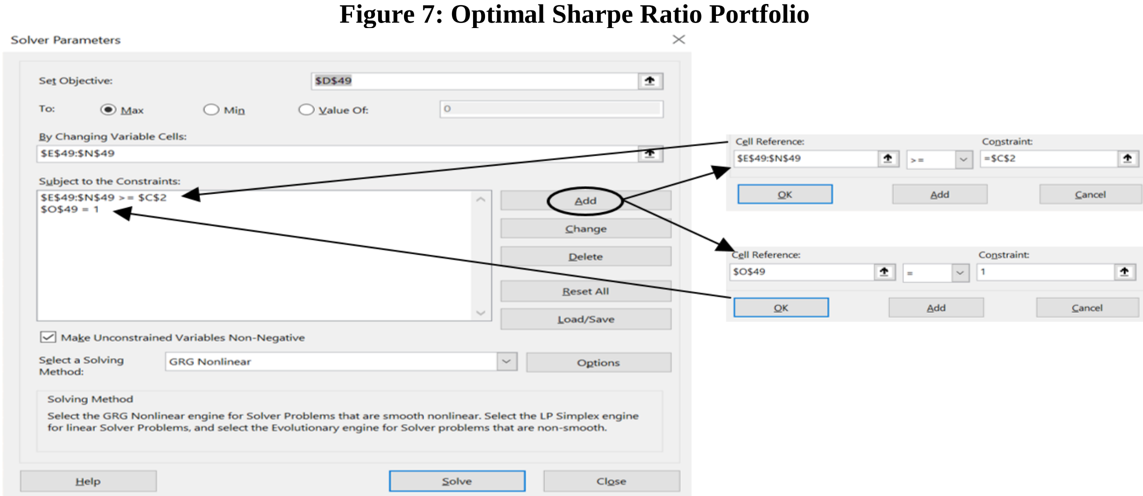
Figure 6: Optimal Return Portfolio



Operation 3

To find the optimal risky portfolio, the optimization “objective” is to “maximize” the portfolio Sharpe ratio by “changing” the weights of each asset class, “subject to the constraints:” (1) all weights add-up to 100 percent, and (2) weights are constrained for short-selling restrictions.

Figure 7: Optimal Sharpe Ratio Portfolio



Row 49 is used to determine the market portfolio. Select **Solver** under the **Data** ribbon. In the **Solver Parameters** dialog box, enter all input as shown in Figure 7:

- Press the **Reset All** button, then press the **OK** button
- Input cell D49 in the **Set objective** box (Shape ratio), and check the **Max** button
- Input E49:N49 (sector weights) in the **By Changing Cells** box
- Under **Subject to the Constraints**:
    - Press the **Add** button. In the dialog box, select O49 as a **Cell Reference**, = in the middle box, type 1 in the **Constraint** box, and press the **OK** button.
    - Press the **Add** button. In the dialog box, select range E49:N49 as a **Cell Reference**, >= in the middle box, cell C2 as a **Constraint**, and press the **OK** button.

- Press the **Solve** button. The **Solver Results** dialog box appears. Choose **Keep Solver Solution** to validate. The optimal weights appear in range E49:N49 and the related portfolio return, risk, and Sharpe ratio are automatically computed in range A49:D49.

Finally, press the **Stop Recording** button located under the **Developer** ribbon.

Automation

To view the VBA code for all operations performed in the “Portfolio Risk and Return” section, select the **Visual Basic** icon under the **Developer** ribbon. Once the VBA editor window appears, press **CTRL-R** to make sure the **VBAProject** window is visible on the left-hand side. Then, expand the file by pressing on the **+** icon next to **VBAProject (file name.xlsm)**, expand the **modules** folder by pressing on the **+** icon next to it, and double-click on **module1** to see the VBA code recorded.

The code has 28 command lines. The first and last line start and finish the program. Lines 2 to 9, lines 10 to 19, and lines 20 to 27 refer to the first, second, and third operation described above.

To check if the program works, run it by pressing the play button below the **Debug** menu item. The recorded code replicates the operations recorded, pausing at the end of each of the three optimization processes when the **Solver Results** dialog box appears (press the OK button to allow the program to continue).

As shown in Figure 8, we modify the recorded code by making the following entries and deletions:

- We delete all redundant commands by removing multiple repeating occurrences starting with **SolverOk**.

- We mute the **Solver Result** dialog box: To stop the **Solver Result dialog box** from appearing at the end of each optimization, and asking to **Keep Solver Solution** or **Restore Original Values**, we add **True** after the **SolverSolve** command.

- We insert a code to refresh all Capital IQ® data when launching the program: Starting after line 1, we add four command lines to call the **Refresh Workbook** command in the S&P Capital IQ ribbon.[11] (this step is optional if Capital IQ is not used)

- We loop the process of finding a maximum portfolio return for a given standard deviation (operation 2 in the optimization section):

  - Define i as an integer variable by typing **Dim i as Long** at the beginning of the section coding for operation 2.

  - Loop the process by inserting **For i=21 to 48** in the line following 4.a, and **Next** at the end of the section coding for operation 2.

  - Replace $O$21 by **Cells(i, "O")**, $B$21 by **Cells(i, "B**"), $A$21 by **Cells(i, "A")**, and $E$21:$N$21 by **Intersect(Range("E:N").EntireColumn, Rows(i))**.

- Calibrate each optimization to make sure that each operation outputs a global solution:[12]

  - Insert the command line **Range("E20:N49").Value = "10%"** just after the four Capital IQ command lines; this command automatically allocates a seed value of 10% to each of the ten sectors weights before the start of operation 1.

---

[11] This code is copied from Capital IQ® Developer Handbook.

[12] The solver function is a free, but rather inefficient tool. To minimize the chance of finding local maxima, we must calibrate the solver by using the previous optimization solution to initiate the next optimization.

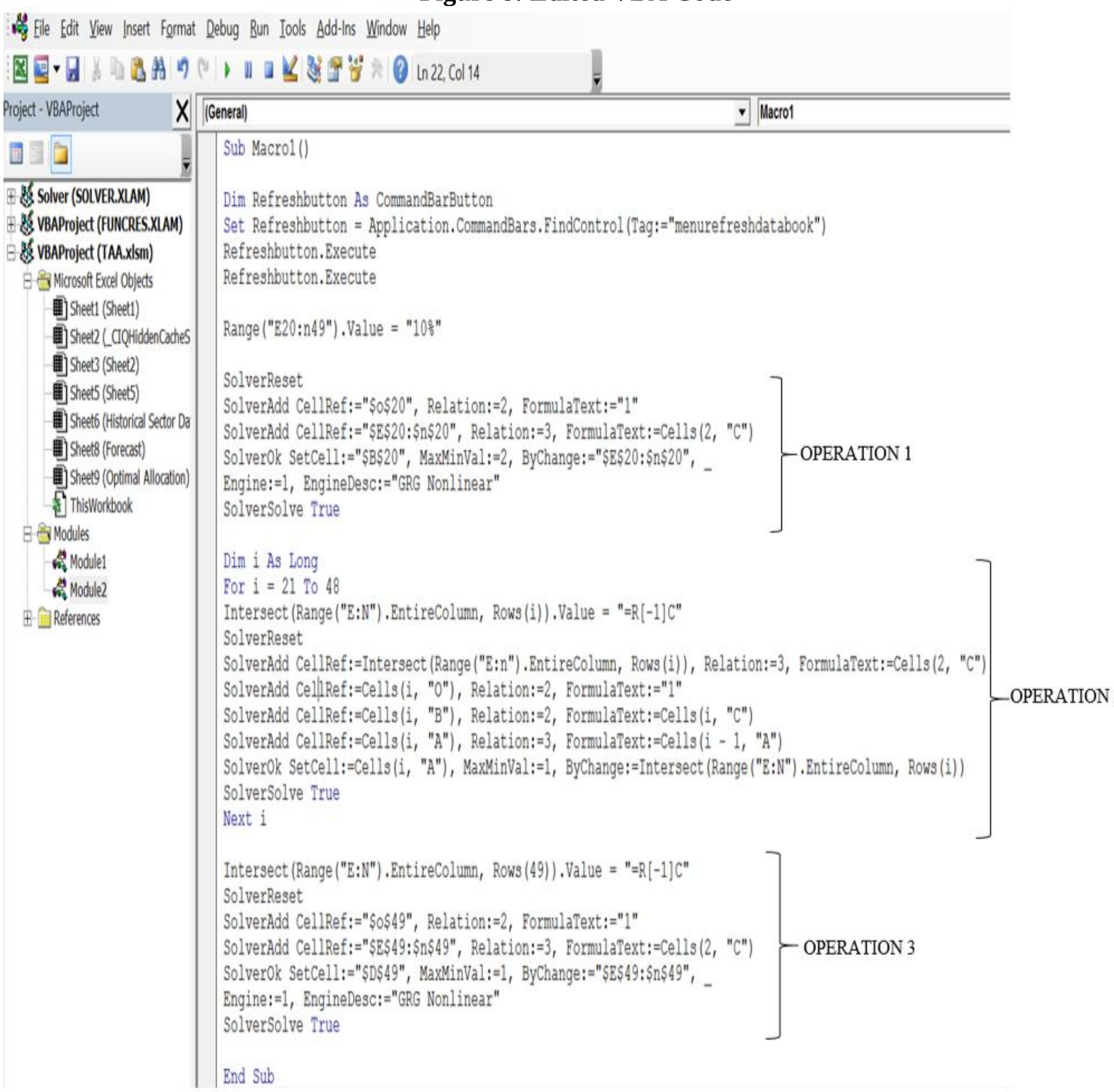
○ Insert the command line **Intersect(Range("E:N").EntireColumn, Rows(i)).Value = "=R[-1]C"** just after the line starting with a **For** statement; this way, we are forcing operation 2 to use the previous (weight) output as a seed value for the next optimization.

○ Towards the end of the section coding for operation 2, insert the command line **SolverAdd CellRef:=Cells(i, "A"), Relation:=3, FormulaText:=Cells(i - 1, "A")** above the **SolverSolve** statement. This command line's purpose is to constrain the optimization output to be economically feasible—i.e., a portfolio return should increase as its standard deviation increases.

○ Insert the command line **Intersect(Range("E:N").EntireColumn, Rows(49)).Value = "=R[-1]C"** at the beginning of the section coding for operation 3--i.e., we use the output E48:N48 to kick off operation 3.

To validate all changes to the initial program, select **Debug** in the **Visual Basic** editor menu, then **Compile** VBA Project in the drop-down menu.

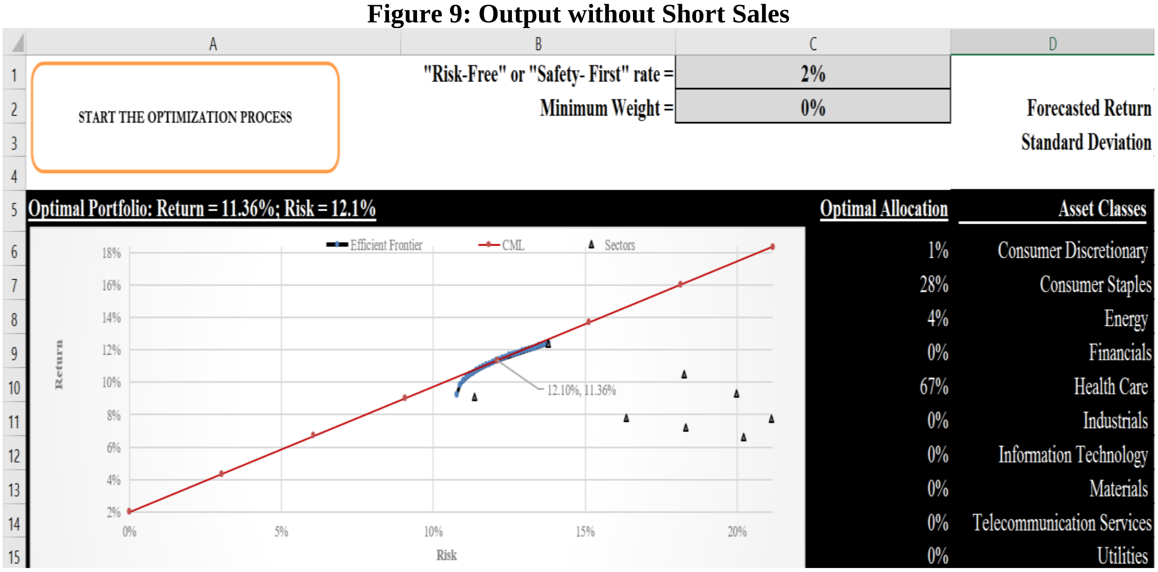
Figure 8: Edited VBA Code



To run the macro from the "optimal allocation" worksheet, right-click on the shape placed on the top left corner (the shape created in section 2.2), select **Assign Macro** in the drop-down menu, choose Macro1 in the

Assign Macro dialog box and validate by pressing the **OK** button. Manually set the values in range C1:C2 and click on the shape. In cell C1, we type the value 2% for a risk-free return. This value is a manual input and can also be tailored to set-up a "safety-first" optimal allocation. Depending on the computer speed, the program runs in approximately fifteen seconds. Assuming a risk-free rate of 2% and no short positions allowed (minimum weight is 0%), you will obtain something that resembles Figure 9. The output range shows the best portfolio coordinates, its composition, and a graph depicting an efficient frontier and inherent CML. As of January 3, 2017, the optimal portfolio had an expected return of 11.36%, a risk of 12.10%, and was composed of approximately 1% in Consumer Staples, 28% in Consumer Staples, 4% in Energy, and 68% in Healthcare. An individual investor would look at range A53:N60 to choose an allocation based on his/her risk preference. For instance, if he can handle 50% more risk than the market portfolio (this is equivalent to a cash weight of -50%), he will borrow $.5 for each of his/her dollar invested. The investor will allocate 1% in Consumer Staples, 42% in Consumer Staples, 6% in Energy, and 101% in Healthcare. Such portfolio has an expected return of 16.0% and a total risk of 18.1%.[13]

Figure 9: Output without Short Sales



There are infinite variations on how to constrain the weight allocation. In this spreadsheet, we can change weight restrictions by modifying the weight constraint minimum value located in cell C2. For instance, we use the spreadsheet to evaluate the impact of weight constraints on the efficient frontiers and CML. In Figure 10, we consider three cases where the weight constraints are 5% (at least 5% in each sector), 0% (only long positions in each sector), and -10% (for each sector, a maximum of 10% of the total investment can be sold short). Consistent with the underlying theory, it is clear that more restrictions (5% weight constraint) on the investment mix provides less reward-to-risk than fewer restrictions (-10% weight constraint). It is a critical step in the lecture about modern portfolio theory and how it deals with constraining the investment universe. Discussions about so-called socially responsible investments usually follow.

Concluding Remarks

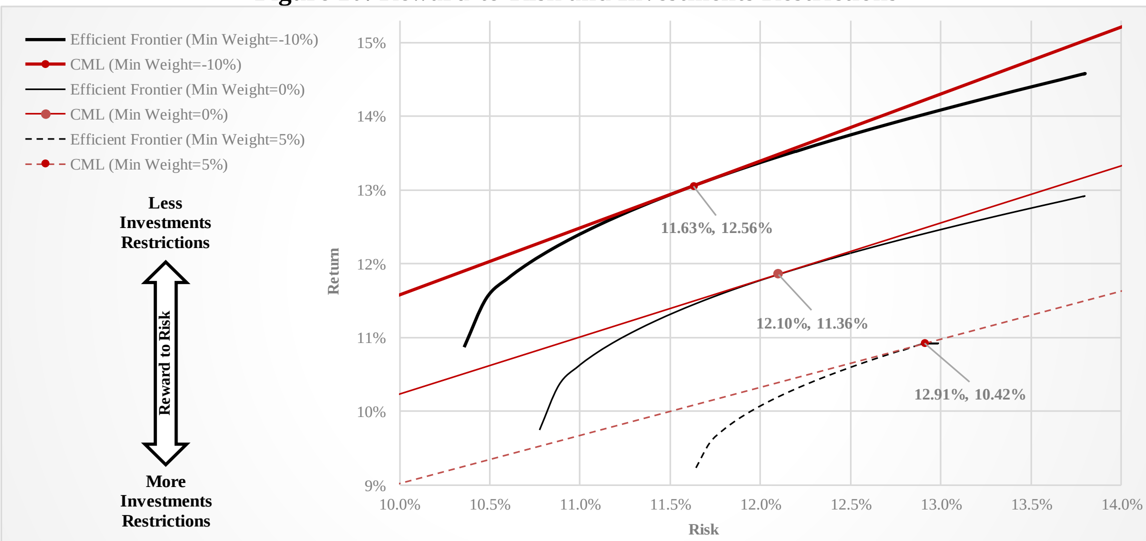
The paper shows how to use VBA to automate the building of the efficient frontier in the context of a tactical sector allocation strategy. For many years, we have been using this example (or any of its variants—e.g., regional, country, industry, or sub-industry tactical asset allocation) as one of the many trading room assignments required to pass our undergraduate investments course. Our students complete this assignment by watching an instructional video that uses most of this paper as a script.

Our students' response to the use of VBA to automate a financial model has been excellent. They like what they can do with it, and enjoy doing things they could not imagine being able to do. It is important to mention that very few of them have had a structured programming course earlier; since VBA is

[13] Usually, this is the time when instructors are rewarded with that "aha moment" clearly noticeable on students' faces.

straightforward, it is an excellent first step towards learning to program. Furthermore, in the case of better students, learning VBA can provide a pathway to interests in more advanced structure programming.

Figure 10: Reward-to-Risk and Investments Restrictions



Ten years ago, Bauer wrote that VBA would not be around "in its present form for another 20 years (and perhaps much sooner than that)" (Bauer 2006, p. 62). The reality is that VBA has not changed much in 10 years (at least from the end-user standpoint) and the financial industry frequently utilizes its capability since the solution to a financial problem is similar to creating a good and orderly automatized spreadsheet—i.e., visualize, implement, debug, and systematize. Further, complicated financial engineering computations mix well with VBA used in conjunction with Excel.

VBA allows finance students with no or very little programming knowledge to build programs using Excel's rich collection of functions. In fact, one does not need to know VBA language to program in VBA since all operations can be recorded; and rarely do users wind up with impossible-to-debug "spaghetti code." The process is that natural: simple operations are recorded into a block of codes, and multiple blocks are used to construct complex computational finance models. We view computational finance problems as puzzles where each piece is a set of VBA codes. In essence, we are teaching students how to build puzzles, and this learning process contributes to their ability to think out-of-the-box.

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APPENDIX

A.1 Historical Sector Data

The “Historical Sector Data” worksheet’s construction is broken into three sections: Column A contains the dates, sector indices values are downloaded in columns C-L, and sector returns are computed in columns M-V.

Figure A1: Historical Data Worksheet’s Input

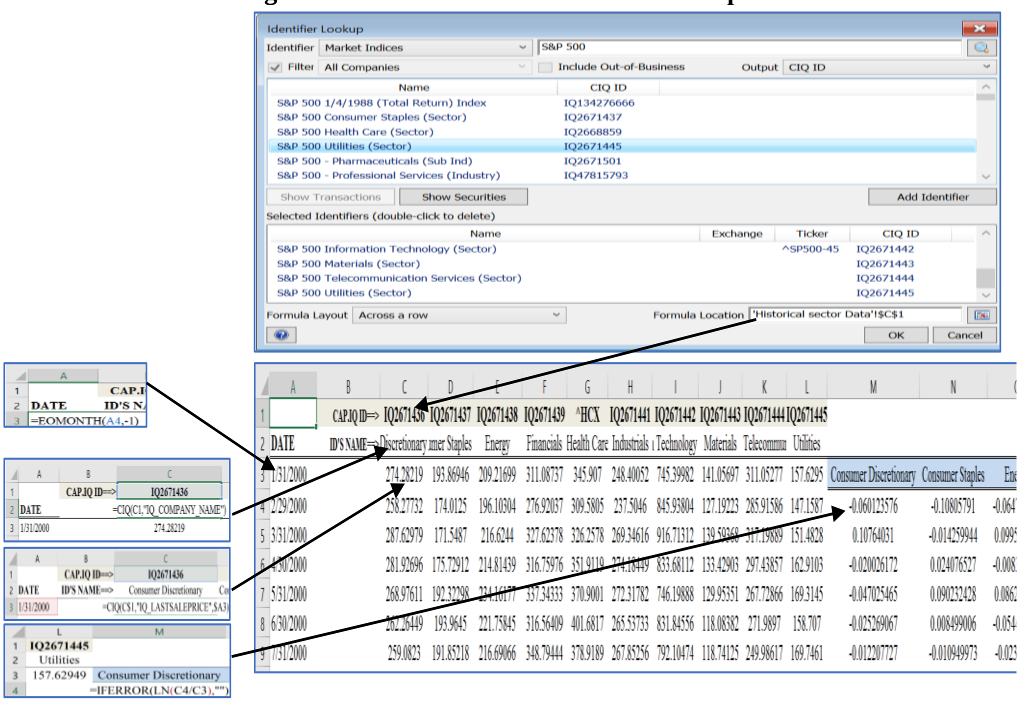

Figure A1 shows how to populate the “historical sector data” worksheet:

• Starting in cells A207 and A206, we insert formulas to compute the date for the end of the current and prior month—i.e., type **=EOMONTH(TODAY(),0)** in cell 207, and **=EOMONTH(A207,-1)** in cell A206. Then, copy cell A206 and paste it into range A3:A205.

• Retrieve Capital IQ® identifiers for each of the 10 U.S. sectors by typing **S&P 500** in the **Identifier** box located in the **S&P Capital IQ** ribbon and pressing the **ENTER** key. In the **Identifier** dialog box, select **Market Indices** in the top left corner, and press on the search icon in the top right corner. Using the **CTRL** key, select the ten S&P 500 sectors from the list of indices, press the **Add Identifier** button, choose **Across a row** as a formula layout, select cell C1 as the **Formula Location**, and press the **OK** button.

• Convert each identifier number into its name by typing **=CIQ(C$1, "IQ_Company_Name")** in cell C2, copying cell C2, and pasting it into range D2:L2. Then, type **=C2** in cell M3, copy cell M3, and paste it into range N3:V3.

• Download each series price by typing **=CIQ(C$1, "IQ_LastSalePrice",$A3)** in cell C3, copying cell C3, and pasting it into range C3:K207. All values can be updated using the "**Refresh Workbook**" button located on the left-hand side of the **S&P Capital IQ** ribbon.

• Calculate the first series' monthly return by typing **=IFERROR(LN(C4/C3),"")** in cell L4. All series' monthly returns are computed by copying cell M4 and pasting it into range M4:V207.

A.2 Forecasted Return

Periodic returns are computed using target prices and dividends, both estimated from proforma financial statements.[14] Each sector's constituents (stocks) 1-year returns are computed using analysts' mean target price and dividend estimates. Thus, each sector's 1-year forecasted return is an average of its constituents' forecasted returns. We organize the "Forecast" worksheet's construction into three sections: Column A contains the S&P 500 constituents' tickers, return forecast inputs for each constituent are in columns B-E, and each constituent's return forecast computations are in columns F-H.

Figure A2: Constituents' Expected Return Computations

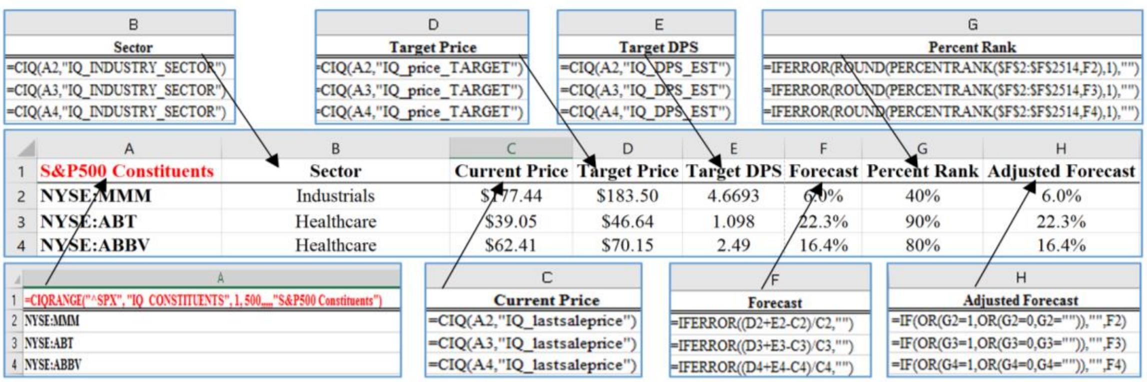

Figure A2 shows how to populate the "forecast" worksheet:

• Download all the constituents of the S&P 500 by typing **=CIQRANGE("^SPX", "IQ_Constituents",1,500,,,,,"S&P 500 Constituents")** in cell A1 and pressing the **ENTER** key.

• Download the sector's name, last price, next twelve month mean target price, and next twelve month dividends per share for each constituent—i.e., type **=CIQ($A2,"IQ_Industry_Sector")** in cell B2, **=CIQ($A2,"IQ_LastSalePrice")** in cell C2, **=CIQ(A2,"IQ_Price_Target")** in cell D2, and **=CIQ(A2,"IQ_DPS_EST")** in cell E2. Then, copy range B2:E2 and paste it into range B3:E501.

• Compute the 1-year return by typing **=IFERROR((D2-C2+E2)/C2,"")** in cell F2, copying cell F2, and

[14] Instructors could use this section to remind students of the two primary approaches to computing expected equity returns, and the implication of each method on strategic versus tactical asset allocation-i.e., computing long-run intertemporal average returns using a factor model, or computing a periodic return using target prices and dividends, both estimated from proforma financial statements. Since tactical asset allocation is a continuous and dynamic process trying to capitalize on short-run relative mispricing, the second approach is preferred. If students have not yet been introduced to asset pricing models, Girard and Ferreira (2004) suggest a naïve forecast based on historical return and a "guesstimate" of what the long run returns should be. Indeed, it is important to emphasize that expected returns should be used.

pasting it into range F3:F501.

• Eliminate outliers by only including forecasts between the 5th and 95th percentiles. Type **=IFERROR(ROUND(PERCENTRANK(F2:F501,F2),1),””)** in cell G2 and **=IF(or(G2=1,OR(G2=0,G2=””),””,F2)** in cell H2; copy range G2:H2 and paste it into range G3:H501.

A.3 Sectors' Return, Risk, and Covariance

To compute each sector's 1-year forecasted returns, type **=AVERAGEIF(Forecast!$B$2:$B$501, E1,Forecast!$H$2:$H$501)** in cell E2 as shown in Figure A3, copy **cell** E2 and paste it into range F2:N2.

Figure A3: Sectors' Forecasted Return, Standard Deviation, and Covariance Matrix

	D	E	F	G
1		Consumer Discretionary	Consumer Staples	Energy
2	Forecasted Return	=AVERAGEIF(Forecast!$B$2:$B$503,E1,Forecast!$H$2:$H$503)	=AVERAGEIF(Forecast!$B$2:$B$503,F1,Forecast!$H$2:$H$503)	=AVERAGEIF(Fo
3	Standard Deviation	=(E6*12)^0.5	=(F7*12)^0.5	=(G8*12)^0.5
4				
5	Asset Classes	Consumer Discretionary	Consumer Staples	Energy
6	Consumer Discretionary	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
7	Consumer Staples	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
8	Energy	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
9	Financials	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
10	Health Care	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
11	Industrials	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
12	Information Technology	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
13	Materials	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
14	Telecommunication Services	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria
15	Utilities	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covariance('Historical Sector Data'!M4:V206)	=Variance_Covaria

There are several ways to create a covariance matrix. Since Excel's **Data Analysis** is static and only computes populations' pairwise covariances,[15] we create the following function in VBA to build a dynamic "large-sample" covariance matrix—i.e.,

```
Function Variance_Covariance(DATA As Range) As Variant
Dim i As Integer
Dim j As Integer
Dim COL As Integer
Dim COV() As Double
COL = DATA.Columns.Count
ReDim COV(COL - 1, COL - 1)
For i = 1 To COL
For j = 1 To COL
COV(i - 1, j - 1) = Application.WorksheetFunction.Covariance_S(DATA.Columns(i), DATA.Columns(j))
Next j
Next i
Variance_Covariance = COV
End Function
```

To use this function, follow these two steps:

1. Under the **Developer** ribbon, open the **Visual Basic** editor (the first button starting from the left). In the editor's menu bar select **Insert** then **Module**. Copy the code above and paste it into the module, and

[15] A simpler but time-consuming method consists of using the function **COVARIANCE.S(Array 1, Array 2)** and computing each pairwise covariance matrix, one cell at a time.

select **Compile VBAProject** under **Debug** (it is the sixth choice starting from the left in the editor's menu bar).

2. Starting from cell E6, select range E6:N1, and type **=Variance_Covariance('Historical Sector Data'!M4:V206)**, then press the keys **CTRL-SHIFT-ENTER** simultaneously (start with the **CTRL** key, then **SHIFT**, then **ENTER**).

Since the covariance between a series and itself is equal to the series' variance, each sector's standard deviation is computed using the square root of the trace in the covariance matrix. Further, we multiply the periodic standard deviations by the square root of the data frequency to annualize them—i.e., $12^{½}$ for monthly data. As shown in Figure 5, type **=(E6*12)^0.5** in cell E3, **=(F7*12)^0.5** in cell F3, **=(G8*12)^0.5** in cell G3, etc.

Teaching the Economics and Convergence of the Binomial and the Black-Scholes Option Pricing Formulas

James R. Garven and James I. Hilliard[1]

ABSTRACT

This paper simplifies the economics of option pricing formulas by clarifying how the no-arbitrage principle ensures that a risk-neutral valuation relationship (based on risk-neutral probabilities) exists between an option and its underlying asset. A spreadsheet exercise shows how binomial probabilities and prices numerically converge to Black-Scholes probabilities and prices, and further numerical analysis reveals how the histogram of terminal stock returns in the multi-period binomial tree converges in probability to the normal distribution. Recommendations for teaching option pricing and convergence include the use of a hypothetical case study of a graduating student's comparison of competing salary offers.

Introduction

It is often challenging for students of finance to grasp fully the logic of the economics and convergence of the binomial and Black-Scholes (1973) option pricing formulas. A rigorous comprehension of these formulas is important not only for investment analysis but also for studying corporate finance topics such as real options, agency theory, risk management, managerial compensation, credit risk, and so on. From a practical perspective, options may also make up an important aspect of future compensation packages for finance graduates. If students understand option pricing theory, they will be better prepared to succeed in their vocational pursuits and personal financial decisions.

In this paper, we provide a pedagogical framework that introduces the basic concepts necessary to understand the economics behind the binomial and Black-Scholes option pricing formulas, and also explains the convergence from the binomial model to the Black-Scholes model. We provide suggestions for walking students through the mathematical portions, and a simple case study example in which students use the binomial and Black-Scholes pricing formulas to evaluate competing salary offers. Finally, we provide a spreadsheet template showing how multi-period binomial model probabilities and prices numerically converge to their Black-Scholes counterparts,[2] and we numerically illustrate how the histogram of the terminal stock return in the multi-period binomial tree numerically converges in probability to the normal density function.

To motivate class discussion, suppose a student receives two competing job offers, and wishes to determine which offer is more financially attractive. Company A has offered a fixed annual salary of $60,000, whereas Company B's offer is for a fixed annual salary of $50,000 plus an employee stock option (ESO) grant for 5,000 shares of Company B's stock, expiring in one year with a $60 per share exercise price. Company B's stock trades for $50, and in our initial numerical example, the stock price will either rise to $62.50 or fall to $40 one year from today. The challenge for the student is to estimate the values of each offer. In subsequent iterations of this example, we replace the binomial outcomes suggested here with

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[2] This spreadsheet uses only the standard Excel functions without relying on macros or other challenging coding techniques.

outcomes based on the volatility of Company B's stock, and the time to expiration is extended beyond one year. We assume throughout the paper that options are European (i.e., exercise may only occur at expiration), and that the underlying asset does not pay dividends.[3]

In the next section of the paper, we feature the single-period versions of the delta hedging and replicating portfolio approaches to pricing options, and show how both methods encompass the risk-neutral valuation approach.[4] All three methods rely on the so-called "no-arbitrage" principle, where arbitrage refers to the opportunity to earn riskless profits by taking advantage of price differences between virtually identical investments; i.e., arbitrage represents the financial equivalent of a "free lunch." However, since competition dissipates the opportunity to earn riskless profits, so-called "arbitrage-free" prices for options emerge.

In the section titled "The Multi-Period Model," we extend the risk-neutral valuation model to two or more periods, and show how it generalizes as the Cox-Ross-Rubinstein (Cox et al. 1979) binomial option pricing formula. In the penultimate section of the paper, we illustrate how probabilities and prices under the Cox-Ross-Rubinstein (CRR) model converge to Black-Scholes option pricing model probabilities and prices, and how terminal stock returns in the multi-period binomial tree numerically converge in probability to the normal density function. We provide concluding remarks in the last section of the paper.

The Single-Period Model

Single-period option pricing models based upon delta hedging and replicating portfolio approaches appear in many investment textbooks. We review those models here to introduce our notation, and to provide a complete teaching lesson plan that an instructor can use to illustrate the economic principle of risk-neutral valuation and the convergence from the single-period, binomial options to the continuous-time Black-Scholes option pricing model. The purpose of this section is not to produce new or novel insights about the binomial model; rather, it is to set the stage for use of the employee compensation example to illustrate (i) the origin of risk-neutral valuation from the delta hedging and replicating portfolio approaches and (ii) the convergence from the binomial option pricing model to the Black-Scholes option pricing model.

Delta Hedging Approach

Suppose the student initially applies the delta hedging approach to determine the value of the option component of Company B's compensation offer. The current price per share of Company B's stock is S, and one time-step (δt) from now, the stock will assume one of the following two values: $S_u = uS$ or $S_d = dS$, where $u > 1$ and $d < 1$. We assume that S = \$50, u = 1.25, d = .8, δt = 1 (one year), the exercise price K = \$60, and the continuously compounded riskless rate of interest r = 3%. Figure 1 shows the binomial "tree" for the current (known) stock price and also the future (state-contingent) stock prices, and Figure 2 shows the binomial tree for the current (unknown) call option price and also the future (state-contingent) call option prices.

Next, the student forms a "hedge" portfolio comprising a long position in one call option and a short position in Δ shares of stock. This portfolio is called a hedge portfolio because movements in the stock's value hedge, or offset the effect of movements in the call option's value. The current market value of this hedge portfolio is

$$V_H = C - \Delta S = C - \Delta 50. \tag{1}$$

[3] Hull and White (2004) provide technical modifications for the binomial and Black-Scholes option pricing models studied here. They explicitly consider the incremental pricing consequences for employee stock options (ESOs) of vesting periods, the possibility that employees may leave the company during the life of the ESO, and the inability of employees to trade their options. Notwithstanding the practical importance of these issues, a consideration of such unique features of ESOs goes well beyond the scope of this paper. Our primary purpose here is to motivate student interest in studying and understanding the basics of option pricing which are foundational for both the theory and practice of finance.

[4] While leading financial derivatives textbooks by Hull (2015) and McDonald (2013) also emphasize risk-neutral valuation, Hull (pp. 274-280) motivates risk-neutral valuation via the delta hedging approach, whereas McDonald (pp. 293-300) motivates risk-neutral valuation via the replicating portfolio approach. Here, we clarify how the delta hedging and replicating portfolio approaches both represent sufficient conditions for a risk-neutral valuation relationship to exist between an option and its underlying asset.

Figure 1: Single-Period Binomial Tree for Current and Future Stock Prices

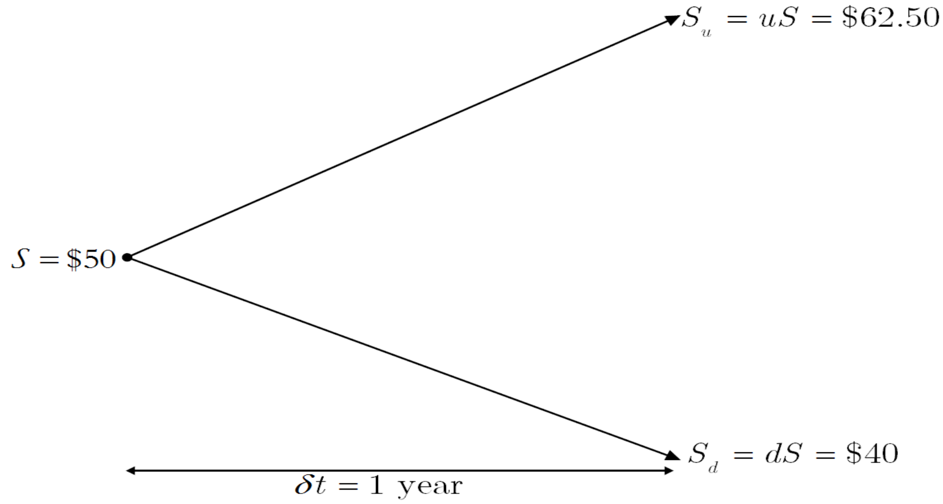
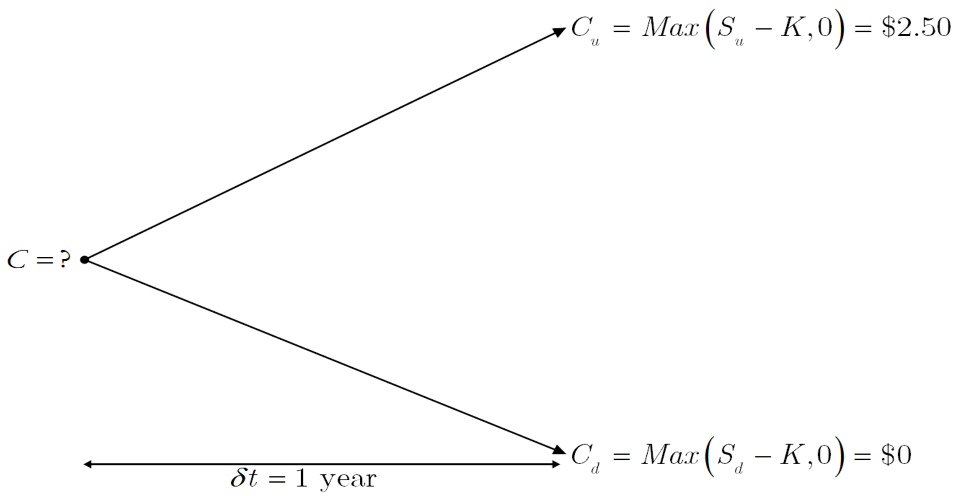


Figure 2: Single-Period Binomial Tree for Current and Future Call Option Prices



At the up (u) node, the value of the hedge portfolio is equal to $V_H^u = C_u - \Delta S_u = 2.50 - \Delta 62.50,$ and at the down (d) node, the value of the hedge portfolio is equal to $V_H^d = C_d - \Delta S_d = 0 - \Delta 40.$ Suppose we solve for Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d.$ Since $V_H^u = V_H^d,$ this implies that $2.50 - \Delta 62.50 = -\Delta 40$ and $\Delta = 0.111.$ Substituting $\Delta = 0.111$ back into the expressions for V_H^u and $V_H^d,$ we find that $V_H^u = V_H^d = -\$4.44.$ An example of this solution for a whiteboard/presentation slide explanation appears in Figure 3. Thus, the terminal value of a riskless hedge portfolio comprising one call option and a short position in one-ninth of a share of stock is equivalent in value to a *short* position in a "synthetic" riskless bond worth \$4.44 one year from now, and the present value of this short bond position is $V_H = -4.44e^{-.03} = -\$4.31.$

Even though the call option and the stock have completely different cash flow characteristics than a riskless bond, the riskless hedge portfolio comprising these two securities creates a "synthetic" riskless bond in the sense that its cash flows mimic the riskless bond cash flows. Under no-arbitrage conditions, the price of the synthetic bond must equal the price of the actual bond with the same payoffs. So, for a given stock price, the price of the call option which satisfies this no-arbitrage condition is the arbitrage-free price. Since

$V_H = C - (0.111)50,$ this implies that the arbitrage-free price for the call option is C = \$1.24, which implies that the proposed option compensation is worth 5,000 x \$1.24, or \$6,200. Since the value of the Company A's \$60,000 salary-only offer exceeds the value of Company B's salary (\$50,000) plus option (\$6,200) offer, our student will prefer Company A's offer, unless the student is risk-loving or assumes a different probability distribution than the one presented here.

Figure 3: Presentation Slide Illustration for Finding Hedge Ratio and Present Value of Hedge

$$C_u - \Delta S_u = C_d - \Delta S_d$$
$$2.50 - \Delta 62.5 = 0 - \Delta 40$$
$$2.50 = \Delta 22.5$$
$$\Delta = \frac{2.50}{22.50} = 0.111 = \frac{1}{9}$$

Then:

$$\begin{aligned} V_H^u &= 2.5 - (0.111)(62.50) \\ &= 2.5 - 6.9438 \\ &= -4.44, \textit{ and} \ldots \\ V_H^d &= 0 - (0.111)(40) \\ &= -4.44 \end{aligned}$$

So:

$$V_H = PV\left(V_H^u\right) = PV\left(V_H^d\right) = e^{-0.03}(-4.444) = -4.31$$

While we would not expect a firm to offer a put option as part of a compensation package to a prospective employee, it is worthwhile to consider how to price an otherwise identical put option with an exercise price of \$60. Since the arbitrage-free price for the call option is \$1.24, we rely upon the put-call parity equation (Stoll 1969) to determine the arbitrage-free price of an otherwise identical put option.

The put-call parity equation is shown in equation (2):

$$C + Ke^{-r\delta t} = P + S. \tag{2}$$

Thus,

$$P = C + Ke^{-r\delta t} - S = \$1.24 + 60e^{-.03} - \$50 = \$9.47. \tag{3}$$

We can also determine the arbitrage-free price for the put option via the delta hedging approach. Since price movements for a put option and its underlying stock are inversely related, we form a hedge portfolio comprising a long position in one put option and a long position in Δ shares of stock. The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 50. \tag{4}$$

At node *u*, the value of the hedge portfolio is equal to $V_H^u = P_u + \Delta S_u = Max(K - 62.50, 0) + \Delta 62.50$ $= 0 + \Delta 62.50,$ and at node *d*, the value of the hedge portfolio is equal to $V_H^d = P_d + \Delta S_d$ $= Max(K - 40, 0) + \Delta 40 = 20 + \Delta 40.$ Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$ implies that $\Delta 62.50 = 20 + \Delta 40;$ thus $\Delta = 0.889.$ Substituting $\Delta = 0.889$ back into the expressions for V_H^u and V_H^d, it follow that $V_H^u = V_H^d = 55.56.$ These calculations can be shown on a whiteboard/presentation slide similar to Figure 3. Thus, the terminal value of a riskless hedge portfolio comprising one put option and a long position in eight-tenths of a share of stock is equivalent in value to a *long* position in a synthetic riskless bond worth \$55.56 one year from now. The present value of this long bond position is $V_H = \$55.56e^{-.03} = \$53.91,$ which implies that P = \$9.47.

In the next section, we explore an alternative approach to option valuation. Rather than infer the value of an option by pricing a synthetic riskless bond, we infer option value by calculating the values of "synthetic" options created with combinations of the underlying stock and a riskless bond.

Replicating Portfolio Approach

Another way for the student to determine the value of the call option is to create a replicating portfolio. Under this trading strategy, the student replicates the call option payoffs at nodes *u* and *d* by purchasing Δ shares of stock today and financing part of this investment by borrowing money. The current market value of the replicating portfolio must equal the current market value of the option; if the replicating portfolio and the option have different market values, the student can earn positive profits with zero risk and zero net investment by buying the less expensive investment and shorting the more expensive one. Thus, we invoke the no-arbitrage condition to establish that the arbitrage-free price of the call option must equal the value of its replicating portfolio.

To replicate the payoffs of the call option, the student forms a hypothetical portfolio comprising Δ shares of stock and \$*B* in riskless bonds. The initial cost of forming such a portfolio is $\$(\Delta S + B)$. When the option expires, its value depends on whether the stock price goes up or down, as shown in equations (5) and (6):

$$C_u = \Delta uS + e^{r\delta t}B, \text{ and} \tag{5}$$

$$C_d = \Delta dS + e^{r\delta t}B. \tag{6}$$

Note that the first term in equation (5) represents the value of the underlying stock at node *u* (uS) multiplied by the number (or fraction) of shares held in the underlying stock. The second term represents the future value of the bond, assuming continuous compounding at the annual rate of *r* during the δt time interval. Equation (6) provides the corresponding value of the replicating portfolio at node *d*. Students will determine how many shares to purchase, and how much to borrow by solving equations (5) and (6) for Δ and *B*:

$$\Delta = \frac{C_u - C_d}{S(u-d)} \geq 0, \text{ and} \tag{7}$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}(u-d)} \leq 0. \tag{8}$$

Note that the equalities in equations (7) and (8) only hold when $C_u = C_d = 0$; i.e., only if the call option always expires out of the money. Otherwise, $\Delta > 0$ and B $<$ 0; i.e., node *u* and *d* call option payoffs correspond to payoffs at these same nodes on a margined investment in the stock based on the Δ and *B* values calculated using equations (7) and (8).

Next, let's reconsider these equations in light of our numerical example. From equations (7) and (8), $\Delta = \frac{C_u - C_d}{S(u-d)} = .111$ and $B = \frac{uC_d - dC_u}{e^{r\delta t}(u-d)} = \frac{1.25(0) - .8(2.5)}{e^{.03}(.45)} = -4.31$. Note that Δ here is the same as the Δ calculated under the delta hedging approach, and the value of *B* is the same as the value of V_H in the earlier approach. These equations can be worked out on a whiteboard/presentation slide as shown in Figure 4. Thus, the student can replicate the call option by purchasing one-ninth of a share of stock for \$5.55 and borrowing \$4.31. Since the value of the replicating portfolio is $\$(\Delta S + B)$ = \$5.55 - 4.31 = \$1.24, this must also be the arbitrage-free value of the call option. Therefore, the decision regarding the choice between Company A's and Company B's compensation offers is exactly the same as the result obtained in the previous section; since Company A's salary-only offer is worth more than Company B's salary plus option compensation package, our student will find Company A's salary offer more financially attractive.

Following similar logic, we can determine the value of the replicating portfolio for the put option. Suppose we form a portfolio comprising Δ shares of stock and \$*B* in riskless bonds. The initial cost of forming such a portfolio is $\$(\Delta S + B)$. At expiration,

$$P_u = \Delta uS + e^{r\delta t}B, \text{ and} \tag{9}$$

$$P_d = \Delta dS + e^{r\delta t}B. \tag{10}$$

Figure 4: Presentation Slide Illustration for Replicating Portfolio Calculations of Δ and B

$$\begin{aligned}\Delta &= \frac{C_u - C_d}{S(u-d)} \\ &= \frac{2.5-0}{50(1.25-0.8)} \\ &= 0.111 \\ B &= \frac{uC_d - dC_u}{e^{r\delta t}(u-d)} \\ &= \frac{1.25(0)-0.8(2.5)}{e^{0.03(1)}(1.25-0.8)} \\ &= \frac{-2}{0.4637} = -4.31\end{aligned}$$

Solving equations (9) and (10) for Δ and *B*, we get:

$$\Delta = \frac{P_u - P_d}{S(u-d)} \leq 0, \text{ and} \tag{11}$$

$$B = \frac{uP_d - dP_u}{e^{r\delta t}(u-d)} \geq 0. \tag{12}$$

Note that the equalities in equations (11) and (12) only hold when $P_u = P_d = 0$; i.e., only if they put option always expires out of the money. Otherwise, $\Delta < 0$ and $B > 0$; i.e., put option payoffs correspond to payoffs at these same nodes on an investment comprising a short position in the stock, coupled with a long position in a riskless bond based on Δ and *B* values calculated using equations (11) and (12).

Next, let's reconsider these equations in light of our numerical example. From equations (11) and (12), $\Delta = \frac{P_u - P_d}{S(u-d)} = -20/22.50 = -.889$ and $B = \frac{uP_d - dP_u}{e^{r\delta t}(u-d)} = \frac{1.25(20)-.8(0)}{e^{.03}(.45)} = \$53.91$. Thus, we can replicate the put option by shorting eight-ninths of a share for \$44.44 and lending \$53.91. Since the value of the replicating portfolio is $\$(\Delta S + B) = -\$44.44 + \$53.91 = \$9.47$, this must also be the arbitrage-free price of the put option.

Although the delta hedging and replicating portfolio approaches to option valuation are motivated differently, both approaches yield the same arbitrage-free prices for call and put options. Note that neither the delta hedging approach nor the replicating portfolio approach require the use of probabilities for calculating option prices. This is a somewhat counter-intuitive result, since one would think the value of an option *should* depend upon the probabilities of up and down movements in the value of the underlying stock. This insight is important as we move forward with one more example of a binomial pricing model approach which relies upon risk-neutral, or risk-adjusted probabilities to calculate arbitrage-free option prices. As we show next, this approach is a logical implication of both the delta hedging and risk-neutral valuation approaches.

Risk-Neutral Valuation Approach

Next, we consider the risk-neutral valuation approach to pricing options. This approach is popular because of its simplicity. However, the most challenging aspect of this approach involves helping students understand where risk-neutral probabilities come from, and what they mean in practice.

In this section of the paper, we have inferred arbitrage-free prices for call and put options by either creating a synthetic riskless bond (via the delta hedging approach) or by creating synthetic call and put options (via the replicating portfolio approach). Investor risk preferences are not a factor when arbitrage-free prices are formed, because we eliminate risk under both trading strategies. Arbitrage-free prices obtain so long as investors take advantage of opportunities to earn riskless arbitrage profits. Therefore, since the valuation relationship between an option and its underlying asset does not depend upon investor risk preferences, we may price options *as if* investors are *risk-neutral*. This idea is a foundational principle for the risk-neutral valuation approach.

We begin our analysis by showing the relationship which exists between the expected return on the underlying stock (μ), the probability of an up move (*p*), and the probability of a down move (1-*p*). Note that

$$E\left(S_{\delta t}\right) = puS + \left(1-p\right)dS = e^{\mu\delta t}S, \tag{13}$$

where $E\left(S_{\delta t}\right)$ corresponds to the expected value of the stock price at expiration and μ corresponds to the annualized expected return on the stock. Solving equation (13) for *p*, we find that

$$p = \frac{\left(e^{\mu\delta t} - d\right)}{\left(u-d\right)}. \tag{14}$$

We present a whiteboard/presentation slide example of solving for μ from equation (14) in Figure 5.

Figure 5: Whiteboard/Presentation Slide for Deriving Required Return Under Risk (μ)

$$p = \frac{e^{\mu\delta t} - d}{u-d}$$

$$p\left(u-d\right) = e^{\mu\delta t} - d$$

$$pu - pd + d = e^{\mu\delta t}$$

$$pu + \left(1-p\right)d = e^{\mu\delta t}$$

$$ln\left[pu + \left(1-p\right)d\right] = \mu\delta t$$

$$\mu = \frac{ln\left[pu + \left(1-p\right)d\right]}{dt}$$

Suppose that investors are *risk-averse* and that the probability of an up move is *p* = 0.60. Solving equation (14) for μ, we find that $\mu = \frac{\ln\left(pu + \left(1-p\right)d\right)}{\delta t} = \frac{\ln\left(.6\left(1.25\right) + \left(.4\right).8\right)}{1} = 6.77\%.$ Given these probabilities and payoffs, risk-averse investors demand an (annualized) expected rate of return on the risky stock that exceeds the riskless rate of interest by 3.77 percentage points. This additional return over and above the riskless rate of interest corresponds to a risk premium that compensates risk-averse investors for bearing risk.

Now suppose that investors are *risk-neutral*. In a risk-neutral market, the expected return on a risky asset is the same as the expected return on a riskless asset, because risk-neutral investors do not demand a risk premium; i.e., $\mu = r.$ Thus, the expected stock price in a risk-neutral market, one time-step from now is:

$$\hat{E}\left(S_{\delta t}\right) = quS + \left(1-q\right)dS = e^{r\delta t}S, \tag{15}$$

where $\hat{E}\left(S_{\delta t}\right)$ corresponds to the risk-neutral expected stock value, *q* corresponds to the risk-neutral probability of an up move, and (1-*q*) corresponds to the risk-neutral probability of a down move. Comparing the right-hand sides of equations (13) and (15), we replace μ with *r* because $\mu = r$ in a risk-neutral market. Solving equation (15) for *q*, we find that

$$q = \frac{e^{r\delta t} - d}{\left(u-d\right)} = \frac{e^{.03} - .8}{\left(.45\right)} = .5121. \tag{16}$$

By using risk-neutral probabilities *q* and 1-*q* rather than risk-averse probabilities *p* and 1-*p*, this ensures that the risk-neutral expected stock value $\hat{E}\left(S_{\delta t}\right)$ will be less than $E\left(S_{\delta t}\right)$ by an amount that corresponds to the dollar value of the risk premium. Since $E(S_{\delta t}) = Se^{\mu\delta t}$ and $\hat{E}(S_{\delta t}) = Se^{r\delta t}$, the dollar value of the risk premium is $E(S_{\delta t}) - \hat{E}(S_{\delta t}) = Se^{(\mu-r)\delta t} = \$50e^{(.0677-.03)1} = \$1.98.$ Because *q* and 1-*q* are rescaled from *p* and 1-*p* in such a way that removes the effect of risk aversion, the initial stock price *S* can be recovered by discounting $\hat{E}\left(S_{\delta t}\right)$ at the riskless rate of interest; i.e., $S = \hat{E}(S_{\delta t})e^{-r\delta t} = (quS + (1-q)dS)e^{-r\delta t} =$

$(.5121(\$62.50)+.4879(\$40))e^{-.03(1)} = (\$51.52).9704 = \$50.$

Next, we calculate the risk-neutral expected values of the call and put option payoffs at expiration by weighting these payoffs by their corresponding risk-neutral probabilities:

$$\hat{E}\left(C_{\delta t}\right) = qC_u + \left(1-q\right)C_d \text{, and} \tag{17}$$

$$\hat{E}\left(P_{\delta t}\right) = qP_u + \left(1-q\right)P_d, \tag{18}$$

where $\hat{E}\left(\cdot\right)$ corresponds to the risk-neutral expected value operator. Here, $\hat{E}\left(C_{\delta t}\right)$ and $\hat{E}\left(P_{\delta t}\right)$ represent the risk-neutral expected values for the call and put option payoffs at expiration. By discounting $\hat{E}\left(C_{\delta t}\right)$ and $\hat{E}\left(P_{\delta t}\right)$ at the riskless rate of interest, we obtain the current arbitrage-free prices for these (single time-step) European call and put options:[5]

$$C = e^{-r\delta t}\hat{E}\left(C_{\delta t}\right) = e^{-r\delta t}\left[qC_u + \left(1-q\right)C_d\right] = e^{-.03}\left[.5121\left(5\right)\right] = \$1.24\text{, and} \tag{19}$$

$$P = e^{-r\delta t}\hat{E}\left(P_{\delta t}\right) = e^{-r\delta t}\left[qP_u + \left(1-q\right)P_d\right] = e^{-.03}\left[.4879\left(20\right)\right] = \$9.47. \tag{20}$$

Since the risk-neutral valuation approach follows as a logical corollary of the delta hedging and replicating portfolio approaches, arbitrage-free prices under risk-neutral valuation must be the same as prices obtained using the delta hedging and replicating portfolio approaches. The decision regarding the choice between the call option or the bonus remains the same as when we created replicating portfolios and synthetic options; since the Company A's salary-only offer is worth more than Company B's salary plus option compensation package, our student will find Company A's compensation offer more financially attractive.

Risk-neutral Valuation and the Delta Hedging Approach

The student may not understand how three different approaches lead to exactly the same conclusion and wishes to better understand the logical connections that exist between the risk-neutral valuation approach and the delta hedging and replicating portfolio approaches. In the next two sections, we show how delta hedging and portfolio replication imply risk-neutral valuation.

Previously, we formed a hedge portfolio comprising a long position in one call option and a short position in Δ shares of stock. At the beginning of the binomial tree, the hedge portfolio value (as shown by equation (1)) is $V_H = C - \Delta S$. Since $\Delta = \dfrac{C_u - C_d}{S\left(u-d\right)}$ (see equation (7)),

$$V_H = C - \Delta S = C - \frac{C_u - C_d}{S\left(u-d\right)}S = C - \frac{C_u - C_d}{\left(u-d\right)}. \tag{21}$$

At expiration, the value of the hedge portfolio will be the same, irrespective of whether the stock moves up or down; i.e., $V_H^u = V_H^d$ implies that $C_u - \dfrac{C_u - C_d}{\left(u-d\right)}u = C_d - \dfrac{C_u - C_d}{\left(u-d\right)}d$. Thus, the arbitrage-free value of the hedge portfolio, V_H, corresponds to the present value of either V_H^u or V_H^d (let's go with V_H^u); i.e., $V_H = C - \dfrac{C_u - C_d}{\left(u-d\right)} = e^{-r\delta t}\left[C_u - \dfrac{C_u - C_d}{\left(u-d\right)}u\right]$ implies that $C = \dfrac{C_u - C_d}{\left(u-d\right)} + e^{-r\delta t}\left[C_u - \dfrac{C_u - C_d}{\left(u-d\right)}u\right]$. Solving for the arbitrage-free price of the call option, we find that

[5] Note that equations (19) and (20) contain equations (17) and (18) respectively, discounted at the riskless rate of interest.

$$\begin{aligned} C &= \frac{C_u - C_d + \left[\left(u-d\right)C_u - uC_u + uC_d\right]e^{-r\delta t}}{u-d} \\ &= \frac{C_u - C_d - dC_u e^{-r\delta t} + uC_d e^{-r\delta t}}{u-d} \\ &= e^{-r\delta t}\left[\frac{e^{r\delta t}-d}{u-d}C_u + \frac{u-e^{r\delta t}}{u-d}C_d\right] \\ &= e^{-r\delta t}\left[qC_u + \left(1-q\right)C_d\right]. \end{aligned} \tag{22}$$

The risk-neutral valuation relationship shown in equation (22) is identical to the risk-neutral valuation relationship shown in equation (19). Thus, the delta hedging approach implies that a risk-neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk-neutral valuation relationship exists between a put option and its underlying stock (cf. equation (20)).

Risk-neutral Valuation and the Replicating Portfolio Approach

Next, we show how the replicating portfolio approach implies risk-neutral valuation. As shown previously, we valued the replicating portfolio as $V_{RP} = \Delta S + B$, where $\Delta = \frac{C_u - C_d}{S\left(u-d\right)}$ and $B = \frac{uC_d - dC_u}{e^{r\delta t}\left(u-d\right)}$ (cf. equations (7) and (8)). Thus,

$$\begin{aligned} C &= \frac{C_u - C_d}{S\left(u-d\right)}S + \frac{uC_d - dC_u}{e^{r\delta t}\left(u-d\right)} \\ &= \frac{e^{r\delta t}\left(C_u - C_d\right) + uC_d - dC_u}{e^{r\delta t}\left(u-d\right)} \\ &= e^{-r\delta t}\,\frac{C_u\left(e^{r\delta t}-d\right) + C_d\left(u-e^{r\delta t}\right)}{\left(u-d\right)}. \end{aligned} \tag{23}$$

Since $q = \frac{e^{r\delta t}-d}{u-d}$ and $1-q = \frac{u-e^{r\delta t}}{u-d}$, substituting q and $1-q$ into the right-hand side of equation (23) yields:

$$C = e^{-r\delta t}\left[qC_u + \left(1-q\right)C_d\right]. \tag{24}$$

Thus, the replicating portfolio approach implies that a risk-neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk-neutral valuation relationship also exists between a put option and its underlying stock (cf. equation (20)).

Now that the logical coherence of the risk-neutral valuation, delta hedging, and replicating portfolio approaches to pricing options in a single-period framework has been shown, our next task involves expanding the risk-neutral valuation model to incorporate multiple periods.

The Multi-Period Model

In the previous section of the paper, we assumed that the student's option-based compensation will expire after a single one-year period. In this section, we expand the model to allow for multiple periods prior to expiration. We will expand the risk-neutral valuation model to two or more periods and then show how it generalizes to the CRR binomial option pricing formula.

Suppose that the student now wishes to determine the value of an otherwise identical call option for 5,000 shares of Company B's stock, expiring after *two* one-year periods. Figure 6 shows the binomial tree for the current and future stock prices at the up (*u*), down (*d*), up-up (*uu*), up-down (*ud*), and down-down (*dd*) nodes, whereas Figure 7 shows the binomial tree for the current and future call option prices at nodes *u*, *d*, *uu*, *ud*, and *dd*. The student will begin at the terminal (*uu*, *ud*, and *dd*) nodes shown in Figure 7, and apply the risk-neutral valuation formula in equation (19) to determine arbitrage-free prices for C_u, C_d, and C. This solution procedure is called "backward induction," since it requires working backwards from the terminal state-contingent values of the call option to the present.

Figure 6: Two-Period Binomial Tree for the Current and Future Shock Prices

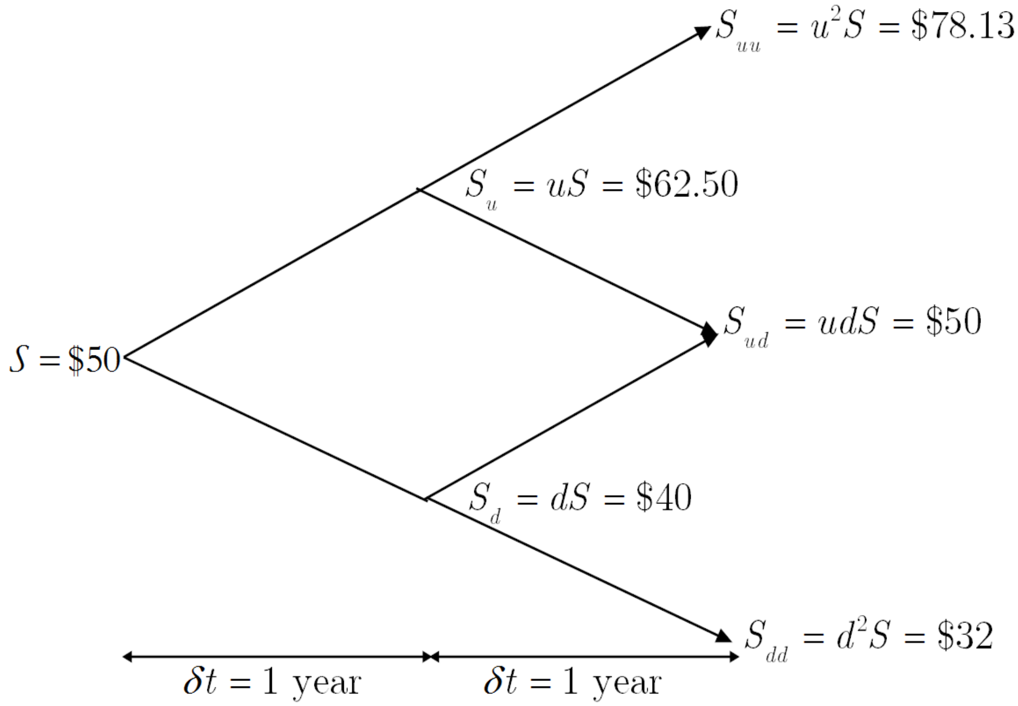
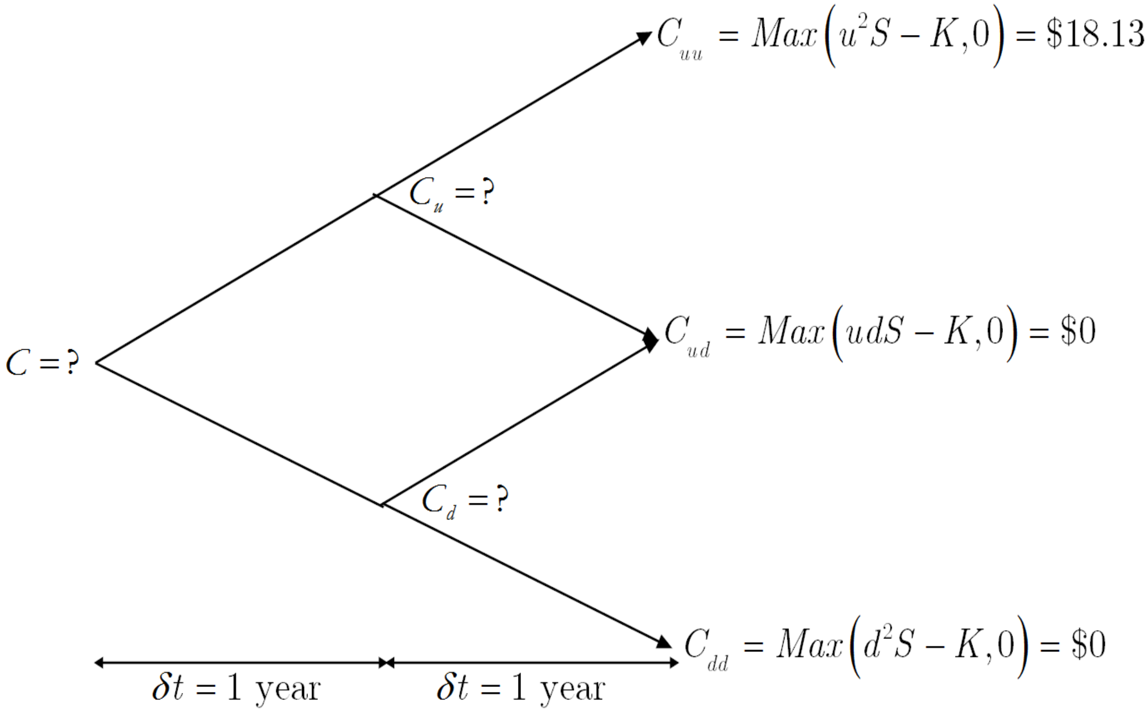


Figure 7: Two-Period Binomial Tree for the Current and Future Call Option Prices



In Figure 7, since the stock only finishes in-the-money at the *uu* node, $C_{uu} = \$78.13 - \$60 = \$18.13$, whereas $C_{ud} = C_{dd} = \$0.$ Thus, the arbitrage-free call option price at node *u* (applying the node *u* version of equation (19)) is $C_u = e^{-r\delta t}\left[qC_{uu} + (1-q)C_{ud}\right] = e^{-.03}\left[.5121(\$18.13)\right]$ = \$9.01. Since $C_{ud} = C_{dd} = \$0,$ it also follows that $C_d = \$0.$ Applying equation (19) once again, the student determines that the current arbitrage-free price of the call option is $C = e^{-r\delta t}\left[qC_u + (1-q)C_d\right] = e^{-.03}\left[.5121(\$9.01)\right] = \$4.48.$ [6] Note that the two-period price is over three times the single-period price of \$1.24. It is well-known that the value of a call option increases as the time to maturity increases. This results from the fact that the underlying asset has more time to increase in value, thus increasing the value of the option if it expires in-the-money. Returning to our compensation example, we can see that an otherwise identical call option expiring in two years rather than one year is now worth \$22,400, making Company B's offer \$12,400 more appealing than Company A's offer.

Although backward induction is required to price the call option via the delta hedging and replicating portfolio approaches, it is unnecessary under risk-neutral valuation. Since the call option included as part of Company B's compensation package is assumed to be European and may only be exercised at expiration, intermediate node prices for the option (such as C_u and C_d) are not needed to find the current arbitrage-free option price (*C*), since the value for *C* depends *solely* on the terminal values of the option. Therefore, the student only needs to undertake the following three steps: 1) calculate the risk-neutral probability for each node at the expiration date, 2) calculate the risk-neutral expected value of the option at expiration, and 3) discount the risk-neutral expected value to present value at the riskless rate of interest for the number of periods to expiration.

The valuation of a multi-period option value (with a few periods) is straightforward for most students. However, understanding that process requires the building blocks shown above (including the delta hedging and replicating portfolio approaches). Once students grasp the basic multi-period risk-neutral valuation model, the next step is to introduce them to the CRR approach to pricing options.

The complexity of analysis grows with each additional time-step. Fortunately, CRR simplify the analysis with their recursive multi-period call option pricing formula, which appears in equation (25):

$$C = e^{-rT}\left[\sum_{j=0}^{n}\binom{n}{j}q^j(1-q)^{n-j}C_j\right]. \tag{25}$$

In equation (25), $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$ indicates how many *j* up and $n-j$ down move path sequences exist in an *n* time-step binomial tree and $T = n\delta t$ corresponds to a fixed expiration date *T* periods from now. Since $q^j(1-q)^{n-j}$ corresponds to the risk-neutral probability of a single *j* up and $n-j$ down move path sequence, the product $\binom{n}{j}q^j(1-q)^{n-j}$ indicates the risk-neutral probability of the stock price ending up at the $j, n-j$ terminal node.[7] C_j corresponds to the payoff on the call option after *n* time-steps and *j* up moves; i.e., $C_j = Max\left[0, u^j d^{n-j}S - K\right].$ The CRR model is considered the canonical binomial option pricing model; besides being the best-known and most cited binomial model, the CRR model also provides a simple matching of volatility with the *u* and *d* parameters.[8] Since $ud = 1.25 \times 0.8 = 1$ in our numerical example, the

[6] Since the two-period call option price is \$4.48, we can determine the price of an otherwise identically configured put option by applying a two-period version of the put-call parity equation given by equation (2); given that $C + Ke^{-r\delta t} = P + S$ for one period, the two-period version of this equation is $C + Ke^{-2r\delta t} = P + S \Rightarrow P = C + Ke^{-2r\delta t} - S \Rightarrow P = \$4.47 + \$60e^{-2(.03)} - \$50 = \$10.98.$

[7] Trivially, the risk-neutral probabilities associated with the *n* + 1 terminal nodes sum to 1; i.e., $\sum_{j=0}^{n}\binom{n}{j}q^j(1-q)^{n-j} = 1.0.$

[8] Specifically, since $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u},$ the variance of stock returns is $\sigma^2\delta t$ (cf. Hull (2015, pp. 286-287)).

CRR model implies that $\sigma = \frac{\ln u}{\sqrt{\delta t}} = .2231$.

Here, we recognize that quantitatively challenged students might struggle with understanding the multi-period CRR call option pricing formula in equation (25). Thus, we suggest an optional, brief tutorial for using summation notation in this problem. Suggested whiteboard/presentation slide content appears in Figure 8. Such students might also appreciate a plain-language reading of equation (25), such as, "The value of a call option is the present value of the weighted average of the values of the call option at expiration, where the weightings represent the risk-neutral probabilities of arriving at each terminal node. Thus, today's call option price is the present value of this weighted average, discounted at the riskless rate of interest."

Figure 8: Presentation Slide Example: Explanation of Equation (25) Summation Notation

Consider the bracketed term in equation (25): $\sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} C_j$,

where n = the number of timesteps, j = the number of up moves to the terminal node, q = risk-neutral probability, and C_j = the value of the call in terminal node j.

The summation symbol tells us to add the simplified expressions for each j starting with 0 until the number of timesteps (2 in our case). So, we will calculate the expression three times (j=0, 1, and 2).

Considering our binomial tree, we know that when j=0 (no up moves, ending in node *dd*), the value of C_j is 0, as the option expires out of the money. The result is similar in our case for j=1 (one up move, ending in node *ud*, which, in a recombining binomial tree, is also node *du*). That leaves j=2 (ending in node *uu*) as the only expression for which we need to simplify the expression.

First, we calculate $\binom{n}{j} = \binom{2}{1}$, which is notation for $\frac{n!}{j!(n-j)!} = \frac{2!}{2!(2-2)!} = 1$. Then, we substitute q, n, j, and C_j into the expression:

$$(1)\left(0.5121^2\right)\left(0.4879^{2-2}\right)(18.13) = 4.75$$

Now, we add the three values of this expression: $0+0+4.75=4.75$ and continue solving the equation.

Suppose n = 1, in which case there is only one time-step and the length of the time-step is $\delta t = T$. Then equation (25) may be rewritten in the following manner:

$$C = e^{-rT}\left[\sum_{j=0}^{1}\binom{1}{j} q^j (1-q)^{1-j} C_j\right] = e^{-rT}\left[(1-q)C_0 + qC_1\right] = e^{-rT}\left[(1-q)C_d + qC_u\right]. \tag{26}$$

Equation (26) is a special case of equation (25), where n = 1. Now suppose that n = 2. Then,

$$\begin{aligned} C &= e^{-rT}\left[\sum_{j=0}^{2}\binom{2}{j} q^j (1-q)^{2-j} C_j\right] = e^{-rT}\left[(1-q)^2 C_0 + 2q(1-q)C_1 + q^2 C_2\right] \\ &= e^{-rT}\left[(1-q)^2 C_{dd} + 2q(1-q)C_{ud} + q^2 C_{uu}\right]. \end{aligned} \tag{27}$$

Equation (25) can be further simplified by rewriting it in such a way which makes it possible to ignore all cases in which the call option is at- or out-of-the-money. However, we need to know the *minimum* number of "up" moves required during n time-steps in order for this to occur. Since the payoff on the call option after n time-steps and j up moves is $C_j = Max(0, u^j d^{n-j} S - K)$, we need to determine the minimum (non-negative) integer value for j such that the call option will expire in-the-money; i.e., so that $u^j d^{n-j} S > K$. Let b represent the *non-integer* value for j such that the value of the underlying asset would be equal to K at expiration; i.e.,

$u^b d^{n-b} S = K.$ Solving this equation for *b*,

$$\begin{aligned} &\ln\left(u^b d^{n-b} S\right) = \ln K \\ &b\ln u + (n-b)\ln d = \ln(K/S); \\ &b\ln(u/d) = \ln\left(K/Sd^n\right); \\ &b = \ln\left(K/Sd^n\right)/\ln(u/d). \end{aligned} \tag{28}$$

Thus, the minimum *integer* value for *j* such that the call option will expire in-the-money is *a*, where *a* is the smallest (non-negative) integer that is greater than *b*. If *a* = 0, this implies that *all the* call option payoffs at the end of the tree are positive. If *a* = *n*, then the only node at which a call option pays off is when there have been *n* consecutive up moves. In theory, *a* can exceed *n*; in that case, the call will always expire out of the money and therefore worthless.

Since $u^j d^{n-j} S - K > 0$ for all $j \geq a$, equation (25) can be re-written as follows:

$$C = SB_1 - Ke^{-rT} B_2, \tag{29}$$

where $B_1 = \left[\sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j} \left(u^j d^{n-j} e^{-rT}\right)\right]$, $B_2 = \left[\sum_{j=a}^{n} \binom{n}{j} q^j (1-q)^{n-j}\right]$, $0 \leq B_1 \leq 1$, and $0 \leq B_2 \leq 1$.

Note that B_1 represents the hedge ratio for the binomial option pricing model and B_2 represents the (risk-neutral) binomial probability that the option will expire in-the-money. Furthermore, SB_1 corresponds to today's value of the underlying asset component of the replicating portfolio, whereas $-Ke^{-rT} B_2$ corresponds to today's value of the margin account used to finance partially the underlying asset component of the replicating portfolio.

Equation (29) resembles the Black-Scholes formula for pricing a European call option. The Black-Scholes formula is given in equation (30):

$$C = SN(d_1) - Ke^{-rT} N(d_2), \tag{30}$$

where $d_1 = \dfrac{\ln(S/K) + \left(r + .5\sigma^2\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and $N(d_1)$ and $N(d_2)$ correspond to the standard normal distribution function evaluated at d_1 and d_2 respectively. Like B_1 and B_2, $N(d_1)$ and $N(d_2)$ are bounded from below at 0 and from above at 1. Note that in the "limiting" case (where $T = n\,\delta t$ remains a fixed value as $n \to \infty$ and $\delta t \to 0$), then B_1 converges in value to $N(d_1)$ and B_2 converges in value to $N(d_2)$. Thus, the interpretations offered in the previous paragraph for B_1, B_2, SB_1, and $-Ke^{-rT} B_2$ also apply to $N(d_1)$, $N(d_2)$, $SN(d_1)$, and $-Ke^{-rT} N(d_2)$.

The convergence of the Cox-Ross-Rubinstein binomial option pricing formula in equation (29) and the Black-Scholes option pricing formula in equation (30) can be shown analytically and numerically. For analytic proofs of how probabilities and prices under the CRR binomial model converge to Black-Scholes probabilities and prices, see Cox et al. (1979) and Hsia (1983). Rendleman and Bartter (1979) independently derive a similar binomial model to that of CRR and provide an analytic proof of the convergence of their model to Black-Scholes in an appendix to their paper. Joshi (2011) also considers various binomial models other than CRR and shows that while the CRR *ud* = 1 assumption is analytically convenient, it is unnecessary to get convergence to Black-Scholes. In the next section of the paper, we will *numerically* illustrate the convergence of the CRR model to the Black-Scholes option pricing model, and leave analytic illustration for graduate-level courses.

Convergence: Numerical

In a spreadsheet model (available at http://bit.ly/options_econ_converge), we numerically illustrate Black-Scholes and CRR model prices based on our employee stock option example in which *S* = \$50, *K* =

\$60, r = 3%, T = 2 years, $\sigma = .2231$, and the option is for 5,000 shares of Company B's stock. Applying the Black-Scholes formula provided in equation (30), we find that $d_1 = \frac{\ln(S/K)+(r+.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(60/50)+(.03+.5(.2231^2)2}{.2231\sqrt{2}} = -.230$, $d_2 = d_1 - \sigma\sqrt{T} = -.230 - .2231\sqrt{2}$ $= -.545$, $N(d_1) = N(-.230) = .409$, and $N(d_2) = N(-.540) = .293$. Thus, the value of a call option to purchase one share of Company B's stock is \$3.91 (as indicated by the Black-Scholes model), and the value of the option component of Company B's compensation offer is \$19,550.

In Table 1, we list CRR model probabilities and prices (based on equation (29)) along with the fixed Black-Scholes model probabilities and price (based on equation (30)) obtained from the spreadsheet model. This table shows that as the number of time-steps increases, the frequency at which the call option expires in-the-money at end-of-tree nodes (as shown by B_2) also varies. The CRR probabilities (as shown in the B_1 and B_2 columns) and CRR model prices swing back and forth as time-steps are added. These swings become less attenuated as the number of time-steps increase, converging toward the Black-Scholes probabilities ($N(d_1)$ = 0.409 and $N(d_2)$ = 0.293) and \$3.91 price. Figure 9 illustrates the convergence in price and Figure 10 illustrates the convergence in probability. Many of the results obtained from our spreadsheet model (including the "sawtooth" image present in Figure 10) are explained in greater detail by Feng and Kwan (2012).

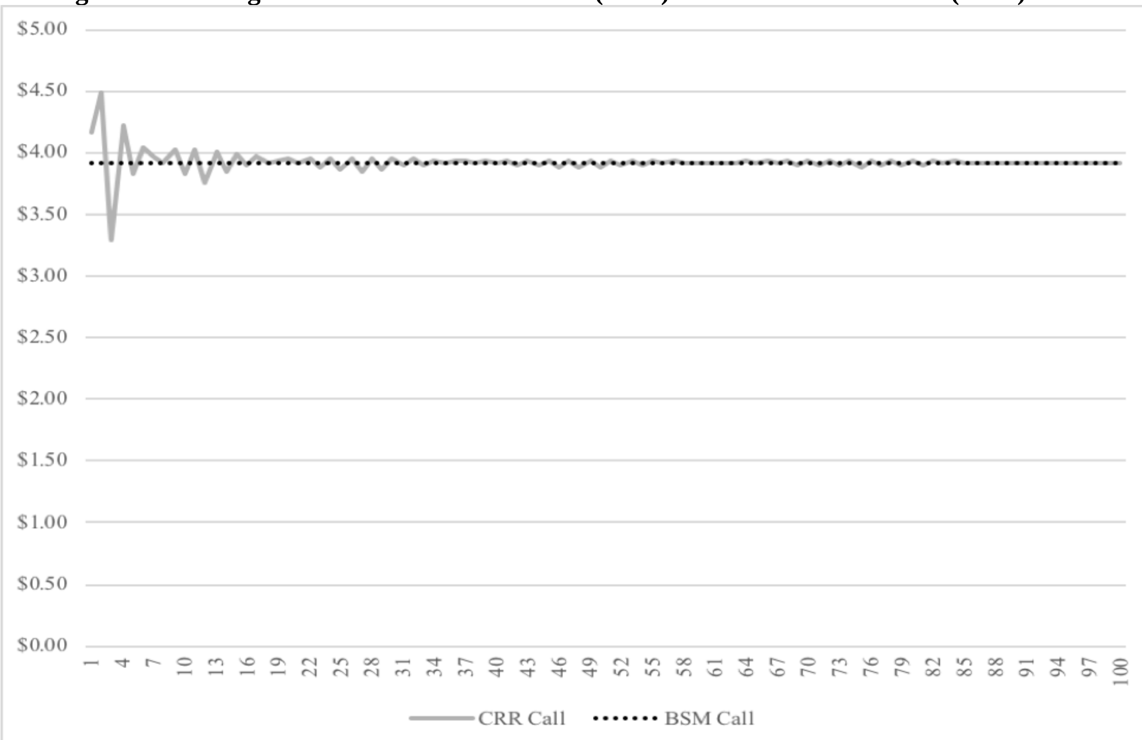
Table 1: Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Option Pricing Model

Time-steps	q	B_1	B_2	CRR Value	$N(d_1)$	$N(d_2)$	Black-Scholes Value
1	0.518	0.669	0.518	\$4.17	0.409	0.293	\$3.91
2	0.512	0.386	0.262	\$4.48	0.409	0.293	\$3.91
3	0.510	0.215	0.132	\$3.29	0.409	0.293	\$3.91
4	0.508	0.452	0.325	\$4.22	0.409	0.293	\$3.91
5	0.507	0.299	0.197	\$3.82	0.409	0.293	\$3.91
10	0.505	0.517	0.390	\$3.83	0.409	0.293	\$3.91
50	0.502	0.360	0.250	\$3.89	0.409	0.293	\$3.91
100	0.502	0.440	0.320	\$3.91	0.409	0.293	\$3.91
200	0.502	0.387	0.273	\$3.91	0.409	0.293	\$3.91
500	0.502	0.408	0.307	\$3.91	0.409	0.293	\$3.91
1000	0.502	0.400	0.285	\$3.91	0.409	0.293	\$3.91
5000	0.502	0.408	0.292	\$3.91	0.409	0.293	\$3.91

Note. —Binomial and Black-Scholes values and risk-neutral probabilities of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r=0.03.

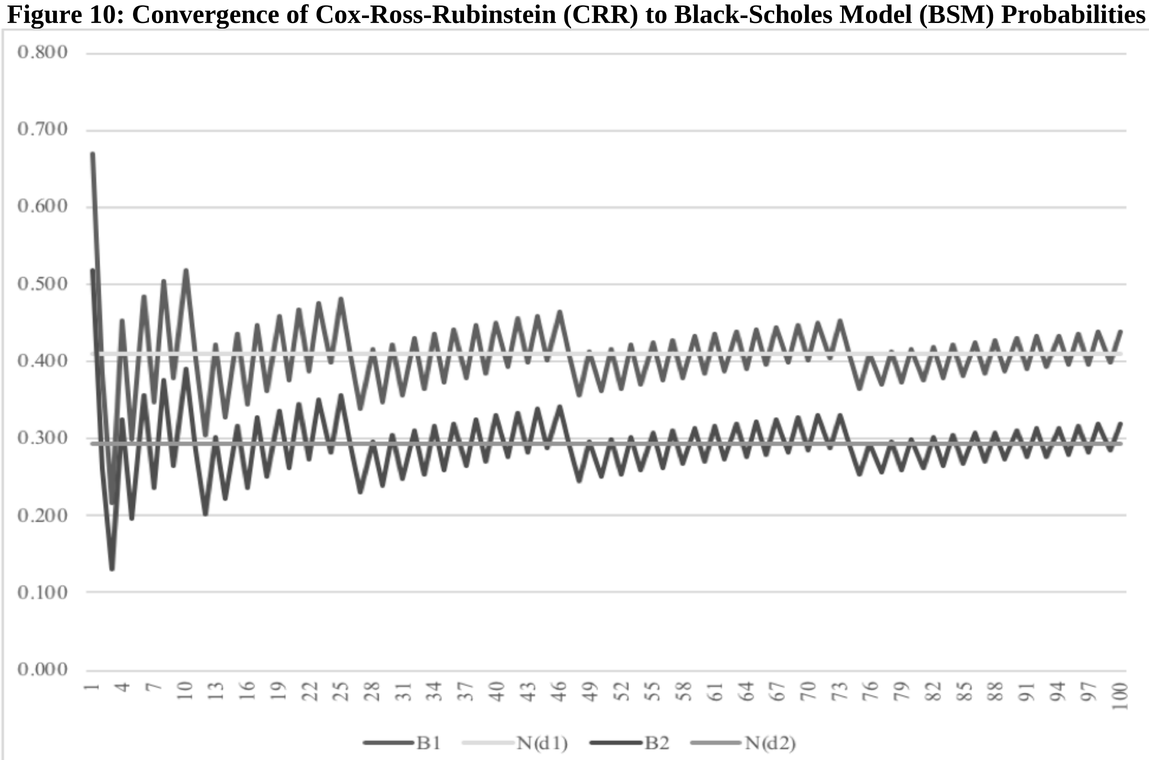
Similarly, we can show that standardized log returns (with a 0 mean and standard deviation of 1) on the underlying asset also converge to the standard normal distribution. Figure 11 shows histograms and corresponding density functions using the same parameter values as in Table 1 and in Figures 9 and 10, while allowing n = {10, 50, 500, and 5,000} and holding $T = n\delta t$ constant. These probability density function charts show convergence from the discrete distribution to the continuous distribution, which follows as a logical consequence of the central limit theorem: as the number of time-steps becomes arbitrarily large, then the discrete distribution converges in probability to the continuous distribution.

Figure 9: Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Model (BSM) Prices



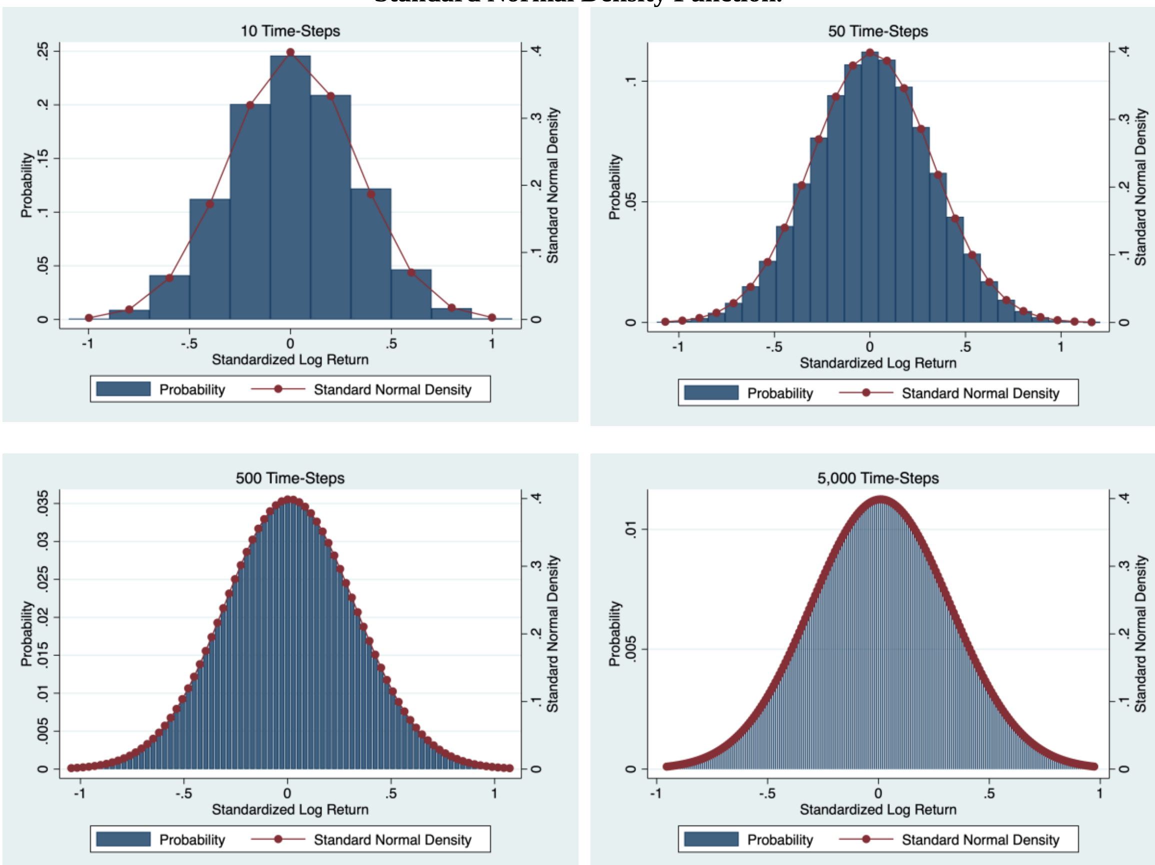
Note.—Binomial and Black-Scholes values of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r=0.03. Number of time-steps represented on the x-axis.

Figure 10: Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Model (BSM) Probabilities



Note.—Binomial and Black-Scholes risk-neutral probabilities of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps represented on the x-axis.

Figure 11: Convergence of Standardized Log Returns under the Binomial Distribution to the Standard Normal Density Function.



Note.—Parameters used: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps are 10, 50, 500, and 5,000.

Conclusion

In this paper, we have provided a simple approach for introducing option pricing models to undergraduate students. We have shown how the delta hedging and replicating portfolio approaches to pricing call and put options imply that risk-neutral valuation relationships exist between option prices and the prices of the underlying assets that they reference. After showing the logical connections between these various approaches in a single-period setting, we show how the risk-neutral approach generalizes to the multi-period case that is captured by the CRR model. Finally, we show how in the limit (as $n \to \infty$ and $\delta t \to 0$ for a fixed time to expiration), 1) the prices and probabilities which comprise the CRR pricing equation in equation (29) converge to the prices and probabilities which comprise the Black-Scholes pricing equation in equation (30), and 2) standardized log returns based upon terminal node prices generated by the CRR pricing equation converge in probability to the standard normal distribution of terminal log returns implied by the Black-Scholes pricing equation.

To further support instruction of option pricing models, we provide some classroom tools, including a limited prospective employee compensation case study,[9] whiteboard/presentation slide examples that can help instructors explain and show the process to their students, and a spreadsheet which shows the convergence between the CRR and Black-Scholes models at http://bit.ly/options_econ_converge.

[9] Note that the non-tradability of employee stock options and various vesting rules provide further opportunities to explore modifications to the Black-Scholes model. These are less tractable than the model presented here, but the principles of convergence are the same.

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Know Math or Take a Bath on a Finance Final Exam

Matthew M. Ross and A. Michelle Wright[1]

ABSTRACT

Math is central to finance education, yet three-quarters of this sample of 159 introductory finance students lack critical quantitative skills on the first day of class, leading to overall underperformance. By utilizing criterion-referenced mathematics pretest items and matching applied finance posttest items, we find that students with substandard math skills rarely catch-up in the quantitative aspects of introductory finance. Indeed, the pretest determines a significant proportion of final exam performance, with the average student gaining a meager 5% between pretest and posttest. We discuss curricular implications of these findings and research-based approaches to facilitate course readiness.

Introduction

Despite the critical role quantitative ability plays in the successful completion of an Association to Advance Collegiate Schools of Business (AACSB 2017) accredited Bachelor of Business Administration (BBA) program, students often struggle with these key prerequisite skills. Employers report that only 28% of recent college graduates are well prepared for the quantitative rigors of the workplace (Hart Research Associates 2015). Yet this inability to perform basic mathematics is not a new problem. Over half a century ago, Wieting (1962, p. 187) observed, "A well-known complaint of employers is that students cannot perform basic arithmetic despite the fact that they have had many years of training in mathematics." As one of the most quantitative business disciplines, finance education researchers have been examining mathematics preparedness for several decades. This study is not the first to inquire how math facilitates acquisition of finance knowledge, but does investigate the problem from a new angle—incoming versus outgoing course skills. Specifically, we examine how students transfer knowledge from a generic form (mathematical skill) to an applied form (finance performance). We begin with assessment of preparedness for quantitative rigor, then measure how specific math skills transfer to matched finance applications, and, finally, investigate how specific math skills generalize to overall finance performance. To the best of the authors' knowledge, this is the first study employing a criterion-referenced assessment of prerequisite quantitative skills on the first day of the course and then matching these same skills to finance application at the course end. We conclude by reviewing motivation research to assist underprepared students and address the administrative challenges of ensuring prerequisite quantitative skills in AACSB accredited finance courses.

Literature Review and Motivation

Given the importance of prerequisite preparedness, especially for a mathematics intensive course like introductory finance, some researchers measure course readiness with a standardized pretest at the start of the term. Grover et al. (2009) link basic mathematics pretest questions to course grades with each correct

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answer on the pretest adding 1.1% to the overall course percentage. Fields (2013) also finds a prerequisite proficiency exam at the beginning of the term is a robust and consistent predictor for final course performance, even after controlling for cumulative college grade point average (GPA) and grades in prerequisite courses. Conversely, Bredthauer and Fendler (2016) find an accounting pretest, but not a mathematics pretest, is a significant determinant of the final course grade in introductory finance. When combined, these results suggest that prerequisites matter and better preparation translates to a greater likelihood of success in introductory finance. We add to this body of literature by examining prerequisite readiness using a criterion-referenced test of ability to manipulate finance equations, administered on the first day of class. Criterion-referenced tests ensure the questions are valid assessments of the topic being tested (Millman 1972). For example, our criterion-referenced math skills question #4 requires solving the equation Y = A + B(X - A) for X to assess the knowledge required for application of a final exam question involving $[E[R_i] = R_{rf} + \beta_i(E[R_m] - R_{rf})]$, the Capital Asset Pricing Model (CAPM). As such, administration of this pretest assessment helps answer our first research question: *Question 1. Is the average student prepared for the quantitative rigors of finance?*

Mathematical ability is a prerequisite for applied financial calculations. Yet, possessing mathematical concepts prior to a course does not guarantee that one can apply these concepts inside the classroom. This concept—transfer of learning (Perkins and Salomon 1994)—is fundamental to education and intuitively appealing: If students learn a concept, they should recognize similar scenarios and apply their knowledge in a slightly modified context. More than a century of research, however, consistently demonstrates that students struggle with transfer of learning (Woodworth and Thorndike 1901; Duncker and Lees 1945; Gick and Holyoak 1983).[2] Difficulty in transfer of learning may stem from inability to recognize similar features across different problem sets (Gick and Holyoak 1980), a change in context between learning and the application (Catrambone and Holyoak 1989), or underdeveloped understanding of the original topic, which unsurprisingly, leads to great difficulty transferring this incomplete knowledge to a new domain (Bransford and Schwartz 1999). Yet, as Caplan (2018, p. 50) emphasizes, even relatively simple knowledge transfer problems can leave individuals stumped, "as a rule, students only learn the material you specifically teach them… if you're lucky." This concern leaves faculty asking: how can I best assist students in transferring math skills to quantitative finance applications?

A more practical concern—a need for clarity in how one defines mastery of introductory finance—further motivates our second research question. Extant research regarding finance prerequisite skills relies on a single dependent variable: course grade at the end of the term. Indeed, of the finance pedagogy research described herein, all studies determine course performance/preparedness by examining final course grades.[3] Respectfully, we suggest that although this data is both relatively easy to collect and correlated with finance skill, final course grades remain an imperfect measure of the ability to solve quantitative finance problems. Some limitations of course grades include the impact of extra credit, homework, attendance, and subjective instructor assessment, which may all reflect effort rather than performance. Given these possible opportunities for bias or subjectivity in final course grade and inspired by Fields' (2013) research utilizing a standardized test, we explicitly link criterion-referenced quantitative skills in a pretest with matching finance application on the final exam. These theoretical and practical concerns motivate our second research question: *Question 2. Do specific math skills transfer to matching finance application?*

Literature is replete with studies demonstrating that many students are underprepared for introductory finance despite passing the quantitative prerequisite courses. Across this research, one variable remains a robust predictor for success in an introductory finance course—cumulative college GPA (Biktimirov and Armstrong 2015). Numerous studies report GPA as a primary determinant of finance course success (Blaylock and Lacewell 2008; Borde et al. 1998; Bredthauer and Fendler 2016; Didia and Hasnat 1998; Nofsinger and Petry 1999; Sen et al. 1997; Simpson and Sumrall 1979). Of course, motive also plays a role with Terry (2002) and Simpson and Sumrall (1979) finding major to be an important factor, with those who self-selected into finance and accounting having higher introductory finance course grades than other business majors. Ely and Hittle (1990) also explore the role of major and find finance majors outperform other business majors in both a managerial economics course and fundamentals of finance course. Finance and accounting majors report that introductory finance is interesting and useful for their future career,

[2] See Barnett and Ceci (2002) for a detailed review of transfer of learning research.

[3] Note we find two exceptions: Bredthauer and Fendler (2016) use final exam grade for one and final course grade for another set of analyses. Carpenter et al. (1993) use expected final course grade, final course grade, and course withdrawal rates in their analyses.

whereas other business majors report that the course is too difficult and only taken because it is required (Balachandran and Skully 2004; Krishnan et al. 1999; Sen et al. 1997). Ely and Hittle (1990) find that earned business credit hours positively and significantly relate to basic finance course performance. Similarly, Blaylock and Lacewell (2008) find a positive relationship between the number of mathematics courses taken and the entry-level finance course grade. This literature motivates our use of GPA, major, and credit hours earned as predictors for finance performance.

The role of demographic variables in introductory finance performance is mixed. With regard to sex differences, some studies report that males tend to outperform females (Borde et al. 1998; Terry 2002), while Sen et al. (1997) report females outperform males, and Didia and Hasnat (1998) report that males and females perform at similar levels. Age is also an uncertain determinant of prerequisite preparedness for finance. Simpson and Sumrall (1979) report older students have lower grades than their younger peers do, while Baloglu and Kocak (2006) find older students have greater math anxiety.[4] Conversely, both Borde et al. (1998) and Terry (2002) find no relationship between student age and performance in introductory finance. To the best of our knowledge, no study explicitly examines the relationship between minority status and quantitative readiness for finance. However, minority students typically do not have the same access to advanced mathematics courses in high school (e.g., calculus) when compared to their non-minority peers (U.S. Department of Education 2014). In addition, Carpenter et al. (1993) find that minority students underperform in a principles of accounting course, which typically serves as a prerequisite for introductory finance. Given the prior research linking introductory finance outcomes with sex, age, and minority status, we control for these demographic variables.

Finally, as finance course goals typically include knowledge acquisition and growth in critical thinking skills, we assess the relationship between the pretest and the cumulative final exam at the end of the term. We believe this is a relevant area of research as it points towards the perennial question of introductory finance instructors and our third research question: *Question 3. How does specific math ability generalize to overall finance performance?*

Data and Methodology

Participants

The sample consists of 159 undergraduate students enrolled in an introductory finance course at an AACSB accredited public business college. All students completed mandatory prerequisite courses or their equivalent (i.e., a math, statistics, and accounting) prior to enrollment. Potential participants included all 180 students enrolled in this course in the 2016-17 regular-term academic year, but six were not present for data collection and 15 did not complete one or more of the items required for inclusion in the study, as with Ross and Wright (forthcoming). Upon enrollment of these 159 participants, 130 are finance majors (81.76%) and 29 do not list finance as a major (18.24%).[5] Motivated by previous research (e.g., Ely and Hittle 1990; Simpson and Sumrall 1979; Terry 2002), we dichotomize participants into finance majors and *Other Major*. The same instructor taught all six of six sections of this undergraduate introductory finance course during consecutive terms of data collection, thereby mitigating selection bias concerns.

Table 1 provides summary statistics and demographic information, along with how the sample compares to the average student enrolled in this AACSB accredited institution. Demographic information (i.e., *Female*, *Minority*, *Age*) and previous academic information (i.e., *Other Major*, *Credits Earned*, *ACT Math*, *GPA*, grade in specific quantitative courses) comes from university records. Study participants show stronger performance at either the 1% or the 5% significance level on every measure of quantitative ability, including *GPA*, *ACT Math* scores, and grades in the prerequisite courses. This comparison suggests that participants have better quantitative skills than the overall business college population at this university.

[4] High anxiety about mathematics is consistently linked with lower quantitative ability (e.g., Ashcraft and Kirk 2001; Ma 1999; McLeod 1994; Yenilmez et al. 2007; Zanakis and Valenzi 1997). Although the exact causal mechanism remains a topic of debate, the inverse relationship between math anxiety and quantitative ability is robust.

[5] Among these 29 *Other Major* participants, 10 list accounting, 11 list management, and eight list dual or other majors. Most (65.52%) of the 29 non-finance majors list a *Finance Minor* while 10 (34.48%) list other minors or do not have a minor designation.

Table 1: Sample Characteristics

	College of Business			Participants			
Variable	*N*	*M*	*SD*	*N*	*M*	*SD*	p^a
Minority	4371	18.03%	38.45%	159	17.61%	38.21%	0.893
Female	4371	35.92%	47.98%	159	22.64%	41.98%	0.001 ***
Other Major	4371	90.80%	28.90%	159	18.24%	38.74%	0.000 ***
Finance Minor	4371	1.05%	10.21%	159	12.58%	33.27%	0.000 ***
Age	4371	23.24	4.32	159	22.44	3.16	0.000 ***
Credits Earned	4371	90.26	22.89	159	85.03	18.29	0.020 **
ACT Math	3328	21.86	4.17	116	23.14	3.96	0.001 ***
GPA	4178	3.10	0.52	150	3.30	0.44	0.000 ***
Math Prereq	2211	2.86	0.78	72	3.16	0.81	0.001 ***
Stat Prereq	2467	3.20	0.74	106	3.38	0.67	0.020 **
Accounting Prereq	2700	2.91	0.70	101	3.11	0.64	0.004 ***

Note. All variables are defined in Appendix B. * $p < .10$. ** $p < .05$. *** $p < .01$.

[a] Unreported Shapiro-Wilk test results indicate non-normality at 1% significance for all College of Business variables. We present the Wilcoxon signed-rank test p-value for robustness.

Procedures and Measures

On the first day of the course, students completed a quantitative assessment where each correct response earned one extra credit point, thereby incentivizing participants to put forth maximum effort on the pretest. Table 2 provides the math assessment score, *Correct Pretest*$_n$, which is the sum of correct responses among *n* problems on the math pretest shown in Appendix A. Each math problem aligns with a finance formula used in popular undergraduate business finance course textbooks (e.g., Berk et al. 2017; Parrino et al. 2017; Ross et al. 2016). A graduate student proctor gave instructions to "Solve for X," with each pretest question yielding one point for a correct answer and zero points for an incorrect answer. Participants answered in an open-ended format with hand-written responses (i.e., the pretest was not multiple choice).

At the end of the course, participants completed a cumulative 50-item multiple-choice final exam, with 15 of the 50 questions requiring a calculation. The final exam, henceforth referred to throughout as exam, results in the *Final Exam* variable ranging from 0 to 100%. Table 2 shows that exam problems 8, 9, 11, 13, 25, 34, 36, and 37 are applications of pretest formulas, yielding *Correct Applied*$_n$ with scores ranging from 0 to *n* points. Individual exam questions are associated with varying degrees of difficulty, varying degrees of instructor focus on the application of specific finance skills, and possibly other difficult-to-measure factors. We present the main findings of this paper setting n = 6, representing pretest questions 1-5 and 10.[6] *Total Gain*$_n$ indicates change over the term of the course as the difference between *Correct Applied*$_n$ and *Correct Pretest*$_n$. *Final Exam* is the score on the exam and *Adjusted Final*$_n$ is the exam score with the *n* applied items removed. Summary statistics for these variables are included in Table 2 and Appendix B provides definitions.

One method of ensuring criterion-referenced materials is to have a group of content experts review the test items for validity (Rovinelli and Hambleton 1977). As such, the math and finance departments assessed each item. The finance department Assurance of Learning Committee rated each quantitative assessment question as a critical prerequisite for success in this particular course ("C"), useful ("U"), or non-essential ("NE"). While all three levels of quantitative skill are required for finance application in this course, useful

[6] The instructor excluded application of pretest items 6 and 7 from the exam after determining these skills to better align with concepts from another introductory level finance course. Items 8 and 9 are potentially problematic due to the Σ and Π notation used to represent sum and product, respectively. Following data collection, the authors learned that this notation was included in only some of the prerequisite math courses.

skills are typically incorporated as part of the instruction and alternative techniques are available to substitute for the non-essential skills, contingent upon mastery of the critical skills.[7] As several math courses satisfy the prerequisites for introductory finance, we had mathematics department faculty members identify the course level where students first encounter the skills required to solve for X. Each rating is included in separate columns in Table 2, with finance faculty ratings under the heading "Rating" and mathematics department ratings under the heading "Prereq."

Table 2: Math Assessment Summary Statistics (n=159)

	Correct Applied $_n$		Correct Pretest $_n$		Total Gain $_n$				
Question(s)[a]	M[b]	SD	M	SD	M	SD	Prereq[c]	Rating[d]	Exam Question
1	0.48	0.50	0.83	0.38	-0.35	0.59	Algebra I	C	11
2	0.33	0.47	0.78	0.42	-0.45	0.60	Algebra I	C	13
3	0.51	0.50	0.22	0.42	0.29	0.61	Algebra II	U	8
4	0.36	0.48	0.35	0.48	0.01	0.60	Algebra II	C	34
5	0.86	0.35	0.33	0.47	0.53	0.56	Algebra II	C	25
6	n/a	n/a	0.13	0.33	n/a	n/a	Algebra II	U	n/a
7	n/a	n/a	0.03	0.18	n/a	n/a	Algebra II	NE	n/a
8	0.63	0.48	0.05	0.22	0.58	0.53	n/a	NE	37
9	0.68	0.47	0.01	0.08	0.67	0.47	n/a	NE	9
10	0.34	0.48	0.07	0.25	0.27	0.49	Algebra II	U	36
1-10	n/a	n/a	2.79	1.75	n/a	n/a			
1-5 & 8-10	4.19	1.79	2.64	1.51	1.55	1.86			
1-5 & 10	2.88	1.41	2.58	1.38	0.30	1.57			

Notes. Following data collection, the instructor determined that items 6 and 7 were more closely aligned with another introductory undergraduate finance course. Therefore, the final exam did not include application of math skills shown in questions 6 and 7.

[a] See Appendix A for equations.

[b] M represents percent correct for each item (e.g., .83 indicates 83% of participants correctly responded to this item).

[c] Prereq provides an assessment from the mathematics faculty of the course in which the skills required to complete the question are first covered.

[d] Rating provides an assessment from the finance faculty of the importance of the skill rated: C-critical, U-useful, and NE-non-essential.

Data Analysis

We employ common statistical techniques including correlation coefficients, weighted regressions, and logistic regressions. We demean all non-dummy independent variables and use the White (1980) adjustment for heteroscedasticity to produce robust statistics. We report the Likelihood Ratio in favor of the Wald statistic in Tables 5a and 5b due to the presence of a non-normal dependent variable (Pawitan 2000). We identify and address both data and methodological limitations, yet acknowledge that these techniques merely mitigate rather than eliminate the issues.

Results

We begin this research asking: *Question 1. Is the average student prepared for the quantitative rigor of finance?* Pretest results in Table 2 indicate that the average student in this sample is notably deficient in the specific math skills relevant to finance calculations. Despite four items receiving the rating of "C," or critical

[7] For example, one can use alternative notation in lieu of the product symbol shown in problem # 9 of Table 2.

to master before enrollment in introductory finance, only 25.79% of the participants correctly answer four or more items of the 10 possible. The average participant correctly responds to 2.58 (*SD* = 1.38) of the six-item pretest, even though all participants earned passing grades or credit for prerequisite courses. We believe that low pretest scores are related to the pop-quiz style, open-ended (i.e., not multiple choice), operations-focused, limited-time format.[8] Specifically, this pretest required that students solve for "X" as opposed to solving for a specific number; as such, participants were not able to employ a calculator to sidestep mathematical operations. Nevertheless, these results reinforce the nearly ubiquitous instructor complaint that business students, even self-selected majors and minors in finance, are typically under-prepared for the quantitative rigor of introductory finance.

Table 3: Correlation Matrix (n =159)

Variable Name	Minority	Female	Other Major	Age	Credits Earned	Math ACT
Minority	1.00 -	-0.01	-0.13 *	-0.04	-0.08	-0.28 ***
Female	-0.01	1.00 -	0.09 -	-0.04	0.03	-0.04
Other Major	-0.13 *	0.09	1.00 -	0.07	0.29 ***	0.22 **
Age	-0.07	0.06	-0.03	1.00 -	0.52 ***	-0.22 **
Credits Earned	-0.09	0.01	0.32 ***	0.27 ***	1.00 -	0.06
Math ACT	-0.29 ***	-0.05	0.21 **	-0.15	0.07	1.00 -
GPA	-0.06	0.14 *	0.09	-0.13	-0.04	0.17 *
Correct Pretest $_6$	-0.06	0.03	0.27 ***	-0.12	0.14 *	0.33 ***
Correct Applied $_6$	-0.13	-0.08	0.11	-0.04	0.21 ***	0.27 ***
Total Gain $_6$	-0.06	-0.10	-0.14 *	0.07	0.06	-0.04
Final Exam	-0.14 *	-0.10	-0.01	0.04	0.19 **	0.33 ***
Adj. Final Exam $_6$	-0.13 *	-0.09	-0.04	0.05	0.16 **	0.30 ***

matrix continued	GPA	Correct Pretest $_6$	Correct Applied $_6$	Total Gain $_6$	Final Exam	Adj. Final Exam $_6$
Minority	-0.03	-0.05	-0.12	-0.06	-0.15 *	-0.15 *
Female	0.17 **	0.05	-0.10	-0.12	-0.09	-0.08
Other Major	0.07	0.23 ***	0.09	-0.17 **	-0.02	-0.05
Age	-0.22 ***	-0.13	-0.11	-0.02	0.04	0.09
Credits Earned	-0.05	0.08	0.15 *	0.03	0.18 **	0.16 **
Math ACT	0.20 **	0.33 ***	0.27 ***	-0.06	0.35 ***	0.32 ***
GPA	1.00 -	0.09	0.30 ***	0.18 **	0.38 ***	0.35 ***
Correct Pretest $_6$	0.10	1.00 -	0.31 ***	-0.58 ***	0.25 ***	0.19 **
Correct Applied $_6$	0.30 ***	0.36 ***	1.00 -	0.56 ***	0.65 ***	0.46 ***
Total Gain $_6$	0.18 **	-0.55 ***	0.57 ***	1.00 -	0.29 ***	0.19 **
Final Exam	0.39 ***	0.27 ***	0.60 ***	0.31 ***	1.00 -	0.97 ***
Adj. Final Exam $_6$	0.37 ***	0.21 ***	0.44 ***	0.20 **	0.98 ***	1.00 -

Notes. Variables are defined in Appendix B. Spearman's is reported above and Pearson's reported below the diagonal. * p <.10. ** p <.05. *** p <.01.

[8] The average pretest correct response rate of 43.00% (i.e. 2.58/6) is lower that related research involving similar student populations of Fields (2013) (71.55% correct pretest score) and Bredthauer and Fendler (2016) (76.50% correct accounting pretest and 65.80% correct math pretest scores).

We address the second research question: *Question 2. Do specific math skills transfer to matching finance application*? in Tables 2, 3, 4, 5a, and 5b. Specifically, *Correct Applied*, in Table 2, suggests little improvement throughout the term, with the average participant correctly responding to 2.88 applied items (*SD* = 1.41) on the exam, which translates to an average posttest score of 48% (i.e., 2.88/6). That the average student shows a *Total Gain* of 0.30 (*SD* = 1.57) or 5% improvement from pretest to application is disconcerting. Table 3 presents correlation coefficients among variables and indicates a positive and significant correlation between students demonstrating mathematical proficiency on the first day of class (i.e., *Correct Pretest*) and students correctly applying the same skills on the exam (i.e., *Correct Applied*) ($r = .31$, $p < .01$).[9] These results suggest that while the average student does improve some, initial quantitative skills show a strong relationship with corresponding application on the exam.

Table 4: Regressions of Correct Applied $_6$

Variable Name	Model 1	Model 2	Model 3	Model 4
Intercept	3.00 ***	3.04 ***	3.04 ***	3.09 ***
	(0.14)	(0.13)	(0.14)	(0.13)
Minority	-0.41	-0.46 *	-0.37	-0.41
	(0.26)	(0.26)	(0.24)	(0.25)
Female	-0.28	-0.53 **	-0.30	-0.52 **
	(0.27)	(0.26)	(0.24)	(0.25)
Other Major	0.11	0.01	-0.17	-0.23
	(0.29)	(0.30)	(0.25)	(0.26)
Age	-0.04	-0.09 **	-0.02	-0.06
	(0.05)	(0.04)	(0.05)	(0.04)
Credits Earned	0.02 **	0.02 ***	0.01 **	0.02 **
	(0.01)	(0.01)	(0.01)	(0.01)
GPA		0.96 ***		0.90 ***
		(0.25)		(0.24)
Correct Pretest $_6$			0.35 ***	0.31 ***
			(0.07)	(0.08)
Observations	159	150	159	150
F-Value	2.42 **	5.86 ***	5.51 ***	8.03 ***
Adjusted R^2 without controls	N/A	0.082	0.127	0.204
Adjusted R^2 with controls	0.043	0.164	0.146	0.248
Adjusted R^2 for controls only	0.043	0.082	0.019	0.044

Notes. Variables are as defined in Appendix B except for the demeaned continuous variables *Age, Credits Earned, GPA,* and *Correct Pretest*. Robust standard error in parenthesis (White 1980). * p <.10. ** p <.05. *** p <.01.

Table 4 provides determinants of the correct application of quantitative skills on the exam posttest questions, *Correct Applied*. Model 1 displays only control variables and indicates that *Credits Earned* is significant at the 5% level. Roughly, 50 additional credits earned typically results in one additional *Correct Applied* item. Model 2 shows that a one-point increase in *GPA* is expected to result in about one additional *Correct Applied* (1% level). Consistent with Carpenter et al. (1993), Model 2 results suggest that the *Minority* control variable may be a marginal determinant of applied performance (10% level). Model 3 results indicate that three *Correct Pretest* responses are associated with one additional *Correct Applied* at the 1% level, robust to inclusion of control variables. Finally, Model 4 includes all predictors for *Correct Applied* and exhibits

[9] We report the Spearman rather than the Pearson correlation due to the ordinal nature of variables. We also note that the significant negative correlation between *Correct Pretest* and *Total Gain* demonstrates reversion to the mean.

the best *F*-value and R^2 of the models given. Controls *Minority*, *Other Major*, and *Age* do not appear to be significant determinants of correct application. Females score significantly lower (5% level) than males, mirroring the findings of Borde et al. (1998) and Terry (2002). Cumulative college *GPA* is also a significant determinant of performance (1% level). Other researchers similarly report that GPA is a significant predictor of finance course performance (e.g., Sen et al. 1997) and pretest performance at the beginning of the term (e.g., Bredthauer and Fendler 2016). However, we show novel findings that a quantitative criterion-referenced assessment, *Correct Pretest*, is a significant predictor of applied performance on the six matching finance application questions. Each correctly answered pretest question translates to 0.31 more *Correct Applied*, significant at the 1% level. Quantitative ability at the beginning of the term is a significant and economically meaningful determinant of quantitative application at the end of the term.

We then apply a more rigorous test using item-level analysis to address our second research question: *Question 2. Do specific math skills transfer to matching finance application*? Whereas Table 4 indicates a positive significant and economically meaningful relationship between the *Correct Pretest* and *Correct Applied*, Table 5a examines results at the item-level (i.e., if a student answers pretest item #4 correctly, does the student correctly answer the corresponding exam question #34?). We control for individual question difficulty level by including dummy variables for each of the six questions. Beginning with Model 1 in Table 5a, controls remain qualitatively consistent with the results given in Table 4. *Minority* appears to be negative with marginal effects, with no significant relationship for *Female*, *Other Major*, or *Age*, while *Credits Earned* is positive and significant at the 1% level. Models 2 and 3 suggest that *GPA* and the pretest are roughly comparable indicators, with each significantly improving correct application of a specific math skill on the exam, 0.79 more applied for one *GPA* point versus 0.74 more applied for one pretest point. Model 4 suggests that *Minority* (*OR* = .70, p <.10), *Female* (*OR* = .64, p <.05) and older students (i.e., *Age*; *OR* = .94, p <.05) are all significantly less likely to transfer mathematical knowledge to the matching applied financial calculations at the end of the term. These results are consistent with Table 4 and previous research examining introductory finance performance (e.g., Borde et al. 1998; Terry, 2002; Sen et al. 1997). Conversely, students with more credits earned are significantly more likely to correctly apply mathematical skills at the item-level by course end (*OR* = 1.02, p <.01). Similarly, *GPA* is a significant determinant for item-level application (*OR* = 2.19, p <.01), suggesting that students with stronger academic records are more likely to correctly answer applied questions at the end of the term. Finally, *Correct Pretest* provides additional support for the importance of mastery of prerequisite skills before the start of the course. With all variables included in Model 4, *Correct Pretest* remains a strong and significant predictor where a correct pretest response doubles the odds of the correct finance application (*OR* = 2.02, p <.01). Students beginning the course with advanced quantitative skills more effectively learn and apply this knowledge to financial calculations. Results from Tables 2 through 5a demonstrate that mathematical preparation is essential for application of introductory finance skills. Inadequate prerequisite quantitative skills preclude knowledge transfer to finance application.

Some math skills may be more important than others for finance. Table 5b decomposes *Correct Pretest* into algebra level (i.e. I or II, in Model 5) and finance importance (i.e. Critical or Useful, in Model 6). Results suggest differential transfer of math skills to finance application with Algebra II (*OR* = 2.18, p <.01) and Critical (*OR* = 1.98, p <.01) skills serving as more reliable indicators than Algebra I (*OR* = 1.76, p <.10) and Useful (*OR* = 2.15, p <.05) skills, respectively. The supremacy of Algebra II and Critical skills aligns with the hierarchal or nested nature of mathematics. Specifically, quantitative skills build upon themselves so mastery of these skills on the first day of class should facilitate greater understanding of the course material. For example, students entering the course with math skills sufficient to handle the CAPM equation (i.e. pretest #4 rated Algebra II and Critical) are able to focus on learning just the “finance” implications rather than trying to learn both the remedial “math” and the “finance” elements simultaneously. Regardless of the causality mechanism, Algebra II and Critical rated skills appear to be meaningful categories as determined by the math and finance faculty assessments, respectively. Nevertheless, the Akaike information criterion (AIC) shows that Model 4 exhibits the best fit of the six models tested in Tables 5a and 5b. We also caution against over-emphasizing the Table 5b results, as breaking down the pretest by the aforementioned categories creates small comparison groups (e.g., the Algebra I and Useful categories each contain only two items).

To the best of our knowledge, this is the first study explicitly demonstrating the degree to which students transfer prerequisite quantitative math skills to quantitative finance application. Unsurprisingly, we find the average student demonstrates improvement throughout the term, consistent with the purpose of teaching—that students leave the course with more knowledge than when they entered. Yet, the small magnitude of this improved application is surprising.[10] We find that the average student exhibits an uninspiring *Total Gain* of 0.30 (as reported in Table 2) between the 43% pretest score and the 48% posttest score. We extend this finding with Tables 4, 5a, and 5b, indicating that students entering the course with stronger quantitative skills more successfully transfer this knowledge to finance by the end of the term. Unfortunately, the majority of the students do not begin the course with strong quantitative skills, and applying underdeveloped skills is difficult.

Table 5a: Logistic Regression of Question-Level Correct Final Exam Application

Variable Name	Model 1			Model 2			Model 3			Model 4		
	β [a]	χ^2 [b]	odds [c]	β [a]	χ^2 [b]	odds [c]	β [a]	χ^2 [b]	odds [c]	β [a]	χ^2 [b]	odds [c]
Minority	-0.33	*	0.72	-0.38	*	0.69	-0.31		0.73	-0.35	*	0.70
	(0.19)			(0.20)			(0.19)			(0.20)		
Female	-0.22		0.80	-0.44	**	0.64	-0.23		0.79	-0.44	**	0.64
	(0.17)			(0.18)			(0.17)			(0.18)		
Other Major	0.09		1.09	0.01		1.01	0.00		1.00	-0.07		0.93
	(0.19)			(0.20)			(0.20)			(0.20)		
Age	-0.04		0.97	-0.07	**	0.93	-0.03		0.97	-0.06	**	0.94
	(0.02)			(0.03)			(0.02)			(0.03)		
Credits Earned	0.01	***	1.01	0.02	***	1.02	0.01	***	1.01	0.02	***	1.02
	(0.00)			(0.00)			(0.00)			(0.00)		
GPA				0.79	***	2.20				0.78	***	2.19
				(0.18)						(0.18)		
Correct Pretest $_{1,i}$							0.74	***	2.11	0.71	***	2.02
							(0.19)			(0.19)		
Other controls [d]	Yes, 6 questions			Yes, 6 questions			Yes, 6 questions			Yes, 6 questions		
AIC [e]	1186			1098			1171			1086		
Likelihood Ratio [f]	158.43	***		173.52	***		175.18	***		187.55	***	
Pseudo R^2 [g]	0.204			0.234			0.224			0.251		
Classification Table [h]	Predicted			Predicted			Predicted			Predicted		
	No	Yes	Total	No	Yes	Total	No	Yes	Total	No	Yes	Total
Incorrect	220	375	595	191	350	541	208	376	584	176	351	527
Correct	121	238	359	118	241	359	120	250	370	117	256	373
Sum	341	613	954	309	591	900	328	626	954	293	607	900

Notes. The dependent variable is a dummy equal to 1 if the answer is correct on the final exam and 0 otherwise. *Correct Pretest* $_{1,i}$ is a dummy variable equal to 1 if the pretest question corresponding to the DV question is correct on the pretest and 0 otherwise. Variables are as defined in Appendix B except for the demeaned continuous variables *Age, Credits Earned, and GPA*. * p <.10. ** p <.05. *** p <.01.

[a] The β column displays regression coefficients with standard error reported underneath in parentheses.

[b] The $\chi 2$ column indicates significance of the Wald statistic evaluated against the Chi-squared distribution.

[c] The odds column reports the odds ratio of each variable.

[d] Dummy variables represent Correct Pretest$_6$ individual questions 1-5 and 10.

[e] Akaike information criterion (AIC) indicates the relative quality of the model.

[f] We report the likelihood ratio test of the global null hypothesis, β=0, instead of the Wald statistic due to non-normality of the dependent variable (Pawitan 2000). The Likelihood Ratio and Wald statistics yield consistent results in all 4 models.

[g] The pseudo R^2 presents the Nagelkerke (1991) rescaled adjustment.

[h] The classification table yields the predicted versus actual classification based on the empirically determined 48.01% (458/954) and 48.00% (432/1900) probabilities of a correct answer.

[10] Note, however, the earlier quote by economist Bryan Caplan (2018, p. 50): "as a rule, students only learn the material you specifically teach them… if you're lucky."

Table 5b: Logistic Regression of Question-Level Correct Final Exam Application

Variable Name	Model 5 β [a]	Model 5 χ^2 [b]	Model 5 odds [c]	Variable Name	Model 6 β [a]	Model 6 χ^2 [b]	Model 6 odds [c]
Minority	-0.35	*	0.71	Minority	-0.35	*	0.70
	(0.20)				(0.20)		
Female	-0.44	**	0.64	Female	-0.44	**	0.64
	(0.18)				(0.19)		
Other Major	-0.07		0.93	Other Major	-0.07		0.93
	(0.20)				(0.20)		
Age	-0.06	**	0.94	Age	-0.06	**	0.94
	(0.03)				(0.03)		
Credits Earned	0.02	***	1.02	Credits Earned	0.02	***	1.02
	(0.00)				(0.00)		
GPA	0.78	***	2.18	GPA	0.78	***	2.19
	(0.18)				(0.18)		
Correct Pretest $_{1, i, Algebra\ I}$	0.57	*	1.76	Correct Pretest $_{1, i, Critical}$	0.68	***	1.98
	(0.32)				(0.22)		
Correct Pretest $_{1, i, Algebra\ II}$	0.78	***	2.18	Correct Pretest $_{1, i, Useful}$	0.76	**	2.15
	(0.24)				(0.36)		
Other controls [d]	Yes, 6 questions			Other controls [d]	Yes, 6 questions		
AIC [e]	1088			AIC [e]	1088		
Likelihood Ratio [f]	187.83	***		Likelihood Ratio [f]	187.59	***	
Pseudo R^2 [g]	0.251			Pseudo R^2 [g]	0.251		

Classification Table [h]	Predicted No	Predicted Yes	Total	Classification Table [h]	Predicted No	Predicted Yes	Total
Incorrect	178	353	531	Incorrect	178	351	529
Correct	115	254	369	Correct	117	254	371
Sum	293	607	900	Sum	295	605	900

Notes. The dependent variable is a dummy equal to 1 if the answer i is correct on the final exam and 0 otherwise. *Correct Pretest* $_{1, i, Algebra\ I}$ is a dummy variable equal to 1 if the pretest question corresponding to the DV question is correct on the pretest and belongs to the Algebra I category and 0 otherwise. Correct Pretest variables for Algebra II, Critical, and Useful are constructed in the same manner. Variables are as defined in Appendix B except for the demeaned continuous variables *Age, Credits Earned, and GPA.* * $p < .10$. ** $p < .05$. *** $p < .01$.

[a] The β column displays regression coefficients with standard error reported underneath in parentheses.

[b] The $\chi 2$ column indicates significance of the Wald statistic evaluated against the Chi-squared distribution.

[c] The odds column reports the odds ratio of each variable.

[d] Dummy variables represent Correct Pretest$_6$ individual questions 1-5 and 10.

[e] Akaike information criterion (AIC) indicates the relative quality of the model.

[f] We report the likelihood ratio test of the global null hypothesis, β=0, instead of the Wald statistic due to non-normality of the dependent variable (Pawitan 2000). The Likelihood Ratio and Wald statistics yield consistent results in all 4 models.

[g] The pseudo R^2 presents the Nagelkerke (1991) rescaled adjustment.

[h] The classification table yields the predicted versus actual classification based on the empirically determined 48.00% (432/1900) probability of a correct answer.

Tables 6a and 6b address our third and final research question: *Question 3. How does specific math ability generalize to overall finance performance?* Successful completion of an introductory finance course typically requires quantitative skill, knowledge of terms and concepts, followed by critical analysis of problems. The average exam performance is 68.30% (*SD* = 12.85) with a 94% maximum and 32% minimum score, similar to Bredthauer and Fendler (2016), the only prior study that explicitly addresses pretest scores and exam scores. It is possible that the strong relationship between pretest and correct application observed in Tables

4, 5a, and 5b is limited to the specific six-item assessment. Perhaps students with subpar quantitative skills compensate with stronger performance on the 70% (i.e., 2*(50 exam questions – 15 quantitative questions)) non-quantitative portion of the final exam.

Table 6a: Regressions of Final Exam

Variable Name	Model 1	Model 2	Model 3	Model 4
Intercept	70.32 ***	71.11 ***	70.69 ***	71.48 ***
	(1.19)	(1.01)	(1.18)	(1.00)
Minority	-4.76 *	-4.25	-4.45 *	-3.80
	(2.62)	(2.94)	(2.68)	(3.04)
Female	-2.96	-5.54 ***	-3.12	-5.45 ***
	(2.43)	(2.11)	(2.30)	(2.01)
Other Major	-2.80	-3.88	-4.93 *	-5.80 **
	(2.84)	(2.81)	(2.57)	(2.44)
Age	-0.09	-0.01	0.07	0.24
	(0.35)	(0.35)	(0.34)	(0.39)
Credits Earned	0.15 ***	0.15 ***	0.12 **	0.11 **
	(0.06)	(0.05)	(0.05)	(0.05)
GPA		12.70 ***		12.23 ***
		(2.69)		(2.68)
Correct Pretest $_6$			2.60 ***	2.53 ***
			(0.67)	(0.68)
Observations	159	150	159	150
F-Value	2.25 *	7.68 ***	4.09 ***	9.12 ***
Adjusted R^2 without controls	N/A	0.150	0.065	0.210
Adjusted R^2 with controls	0.038	0.212	0.105	0.276
Adjusted R^2 for controls only	0.038	0.062	0.040	0.067

Notes. Variables are as defined in Appendix B except for the continuous variables *Age, Credits Earned, GPA,* and *Correct Pretest* which are demeaned. Robust standard error in parenthesis (White 1980). * p <.10. ** p <.05. *** p <.01.

To address this overall skills issue, Table 6a examines determinants of correct responses on the exam. Model 1, which includes only controls, shows minority students with marginally lower exam scores (10% level of significance), and students with more *Credits Earned* score higher on the exam (1% level), consistent with results of Tables 4, 5a, and 5b. Model 2 indicates *GPA* is a strong and significant determinant of exam score (1% level). Similarly, Model 3 shows a positive relationship between *Correct Pretest* score and exam score (1% level). Inclusion of all variables in Model 4 shows *Female, Other Major,* and *Credits Earned* as significant controls. Females, on average, score lower on the exam than their male peers (1% level), consistent with Borde et al. (1998) and Terry (2002). Also consistent with previous research (e.g., Ely and Hittle 1990; Terry 2002), we find that non-finance majors typically score lower on the exam (5% level). Students with more credits earned score higher (5% level), consistent with Ely and Hittle (1990). High *GPA* predicts high exam performance with a 1-point increase in GPA (e.g., from 2.50 to 3.50) typically yielding a 12.23% increase on the exam (1% level).[11] This GPA result is consistent with the aforementioned studies examining cumulative college GPA and introductory finance performance (e.g., Nofsinger and Petry 1999; Sen et al. 1997)—specifically, a student's overall academic record is a significant and economically

[11] Interestingly, the mean 3.42 GPA among the 33 females is higher than the mean 3.26 GPA among the 117 males. However, the difference in cumulative college GPA is not significant at the 10% level.

important determinant of success in introductory finance. High *Correct Pretest* is also predictive of a high exam score as each correct pretest item results in a 2.53% higher score on the exam (1% level).

Table 6b: Regressions of Adjusted Final $_6$

Variable Name	Model 1	Model 2	Model 3	Model 4
Intercept	73.10 ***	73.89 ***	73.41 ***	74.21 ***
	(1.20)	(1.03)	(1.20)	(1.02)
Minority	-4.48	-3.78	-4.22	-3.39
	(2.73)	(3.06)	(2.81)	(3.18)
Female	-2.72	-5.09 **	-2.86	-5.02 **
	(2.44)	(2.16)	(2.35)	(2.10)
Other Major	-3.45	-4.43	-5.22 *	-6.07 **
	(2.85)	(2.77)	(2.69)	(2.51)
Age	-0.01	0.18	0.13	0.40
	(0.33)	(0.37)	(0.33)	(0.40)
Credits Earned	0.13 **	0.12 **	0.11 **	0.09 *
	(0.06)	(0.05)	(0.05)	(0.05)
GPA		12.26 ***		11.86 ***
		(2.93)		(2.93)
Correct Pretest $_6$			2.16 ***	2.16 ***
			(0.69)	(0.70)
Observations	159	150	159	150
F-Value	1.89 *	6.61 ***	3.02 ***	7.32 ***
Adjusted R^2 without controls	N/A	0.134	0.039	0.169
Adjusted R^2 with controls	0.027	0.184	0.071	0.229
Adjusted R^2 for controls only	0.027	0.050	0.033	0.060

Notes. Variables are as defined in Appendix B except for the continuous variables *Age, Credits Earned, GPA,* and *Correct Pretest* which are demeaned. Robust standard error in parenthesis (White 1980). * p <.10. ** p <.05. *** p <.01.

That each correct pretest answer is associated with more than one correct exam answer demonstrates the critical importance of math skill in finance education. Compared to a participant with no correct pretest answers, a participant with six correct pretest answers can expect a 15.18% higher score on the exam, even more than the expected difference between the typical C student versus a B student. However, a more appropriate comparison metric involves a one standard deviation higher GPA (0.44) yielding a 5.38% higher score versus a one standard deviation higher pretest (1.41) yielding a 3.58% higher exam score. Both are substantial effects, but we contend that improving quantitative knowledge relevant to the pretest by one standard deviation is more achievable than improving GPA by one standard deviation. That the exam performance relationship holds even with the “gold standard” determinant of GPA included in the model suggests the centrality of prerequisite mathematical skills to finance application. Even when controlling for cumulative college GPA, incoming quantitative skill yields significant predictive power.

Robustness

As a robustness check, Table 6b examines determinants of exam scores after removing the six specific applied questions. Eliminating these items from the exam score allows assessment of overall introductory finance skills performance, regardless of the student’s ability on the specific posttest items. Specifically, the exam included 15 quantitative items (i.e., required calculations), with six matching applied items. As such, the *Adjusted* $Final_6$ dependent variable includes 9 quantitative items and 35 non-quantitative items. Results remain quantitatively similar and, as indicated in Model 4, *Female*, *Other Major*, *Credits Earned*, *GPA*, and *Correct Pretest* remain significant predictors of the non-criterion referenced questions on the exam. This robustness check affirms that initial math skill generally transfers to better introductory finance exam performance, extending beyond just the specific math skills measured with the six-question criterion-referenced assessment. Additionally, this relationship holds even after controlling for *GPA*, suggesting that the pretest is capturing unique predictive variance in applied finance ability.

To ensure the results are not simply an artifact of our particular construction of the six-item dependent variable, we re-run the analysis using eight items. Results are consistent and most coefficients remain quantitatively similar while all remain qualitatively similar. One of the larger differences is *Total* $Gain_8$ = 1.55 versus *Total* $Gain_6$ = 0.30 due to pretest questions 8 and 9 having low *Correct Pretest* response rates with notably higher *Correct Applied* response rates (see Table 2). Nevertheless, those struggling with mathematical skills on the first day of class typically underperform on the last day of the course, regardless of constructing the *Correct Applied* variable with either a six-item or eight-item measure.

Limitations and Future Directions

Although we take care to address possible confounds and biases, no study is without limitations. First, this study could benefit from a criterion-referenced test with more items. A greater number of test questions would increase statistical power and the breadth of the quantitative skill assessment. The relatively small number of items limits the ability to examine gains throughout the term. Second, collecting data from more participants, across more than one academic year, with multiple instructors, and at different universities may improve generalizability of these results.[12] Third, we utilize a criterion-referenced math skills assessment at the beginning of the term matched with exam questions. However, this pretest is not a nationally recognized standard measure of finance criterion-referenced mathematical ability. Fourth, an extra credit offering on the first day of class may be insufficient to assure maximum effort in completion of the math skills assessment, although including GPA in our regressions may help to mitigate this issue.

Conclusion

Prerequisite quantitative skills exhibit a positive, significant, and robust relationship with introductory finance exam performance. Unfortunately, many students do not enter the course with mastery of the prerequisite quantitative skills (Table 2). The average student only exhibits a 5% (i.e. 0.3/6) greater response rate from pretest to posttest, suggesting that although the average student gains, most do not gain much (Table 2). Not surprisingly, those with stronger initial math skills more effectively apply their knowledge to answer matching finance application questions (Tables 4, 5a, and 5b). Furthermore, stronger initial skills generalize to better finance exam performance (Table 6a), and these results hold even when removing the six quantitative posttest items (Table 6b). We show that along with GPA, prerequisite math skills are a key determinant of performance on an introductory finance exam. Overall, this study demonstrates that insufficient math skill serves as a substantial barrier to quantitative finance performance: if a student lacks the necessary prerequisite skills on the first day of class, he or she is unlikely to master that specific material, or related quantitative finance material, by course end.

Insufficient quantitative skill is neither a new problem nor specific to business. Psychology research offers approaches to motivating students to improve their quantitative skills. First, we suggest that instructors emphasize that basic mathematical ability and problem solving is something employers expect. Many faculty view workplace readiness as an implicit reason for mastering the prerequisites; however, as Fields (2013) notes, students often hear that prerequisites are important because they are required for later course success—not necessarily because employers expect it. Indeed, 55% of undergraduates report being well prepared to work with numbers and statistics, whereas only 28% of employers report that new graduates are adequately

[12] University records vary based on status (e.g., transfer student vs. non-transfer). Specifically, the *ACT Math* score is unavailable for 43 students (27.0% of the sample). *GPA* is unavailable for nine transfers (5.7% of the sample) who are new to the university in the term of data collection. Similarly, many participants earned prerequisite course credit at other institutions so we are missing observations: 87 math (54.7%), 53 statistics (33.3%) and 58 accounting. (36.5%). Since including all variables in a single regression would reduce 159 participants to just 51 (32.1% of the sample), we exclude *ACT Math* and prerequisite course grades from analysis.

prepared (Hart Research Associates 2015). This large expectation gap might be modifiable. Motivation research indicates that specific and explicitly stated expectations about an outcome (e.g., "Employers expect me to know math and will compensate me accordingly with generous pay.") help improve motivation to meet the goal (Lawler and Suttle 1973; Wigfield and Eccles 2000). Emphasis regarding employer expectations might best begin in the prerequisite courses where students work to master foundational skills.

Education institutions might alternatively focus on reducing math anxiety to improve mathematics performance (Park et al. 2014). Given that business students report finance is a very difficult and quantitative course (Krishnan et al. 1999), efforts to reduce math anxiety among business students may be worthwhile. Given our findings of the strong linkage between initial math skills and exam performance, anything that can improve quantitative preparedness—even subjective concerns such as feelings about math—should improve performance in introductory finance. Knowledge of deficiencies prior to the course beginning may also motivate remedial course work or independent study to address the gaps. Finance instructors, versed in the harsh discipline of financial markets, may be inclined to prioritize performance over effort. However, learning research consistently shows that students who focus on "trying hard" and "growing" are more likely to succeed than students focusing on "getting the right answer" and validating their natural intelligence (among others, Grant and Dweck 2003).[13] As effort is a necessary but not a sufficient condition for mastering challenging concepts, students lacking grit may give up early in the semester if concepts seem "hard" or require multiple rounds of trying and failing (Eskreis-Winkler et al. 2014).

Our results demonstrate that approximately three-quarters of undergraduate business students may be underprepared for the quantitative rigor of introductory finance and subsequently gain little quantitative finance skill during the course. Furthermore, and perhaps especially troubling, is that our sample demonstrates greater quantitative ability than the average student at this AACSB institution. Our results are consistent with the well-documented positive relationship between prerequisite mathematical ability and finance course grades (e.g., Bredthauer and Fendler 2016; Fields 2013; Grover et al. 2009), yet we emphasize that course grade serves as an imperfect measure of quantitative finance knowledge. This begs the question—do we want students to leave introductory finance with a "passing grade" or with the ability to apply financial knowledge? While these are not mutually exclusive, in this study 88.7% of the students earned a C or better in the course despite the average student correctly answering only 48% (i.e. 2.88/6) of the applied matching exam items. The implication is that many students arrive with poor prerequisite quantitative skills, fail to master the quantitative aspects of the course, and yet manage to satisfy enough course requirements to enter the more advanced courses. When many students arrive in introductory finance courses underprepared for the quantitative rigor, an instructor faces two undesirable choices: fail many students or pass some students that are underprepared for the quantitative demands of intermediate-level finance courses.

While prerequisite quantitative course restructuring might improve the incoming quantitative skills of introductory finance students, this approach often relies on individuals and departments beyond the influence of finance instructors. Unfortunately, business faculty members may have limited control over the prerequisite courses for introductory finance. Given this limitation in combination with our findings, we endorse Fields' (2013) recommendation to implement a quantitative prerequisite entry test for prospective finance students. We encourage critical review of prerequisite courses and skills tests to ensure that students possess the prerequisite math skill needed to learn and apply finance concepts. AACSB faculty and administrators might consider: who is ultimately responsible when students are not adequately prepared for a course they enroll in—the student, the faculty, or the administration?

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[13] This relationship holds even for quantitative topics (Blackwell et al. 2007).

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APPENDIX A: Math Pretest with Solutions and Examples[14]

Solve the following for X:

1) $Y = A + B - X$

$$X = A + B - Y$$

Example: Find taxes paid given operating cash flow, earnings before interest and taxes (EBIT), and depreciation. [OCF = EBIT + DEP – Taxes]

2) $Y = XAB$

$$X = \frac{Y}{AB}$$

Example: Find profit margin given return on equity, total asset turnover, and the equity multiplier. [$\mathrm{ROE} = \mathrm{PM} \times \mathrm{TAT} \times \mathrm{EM}$]

3) $(Y + A) = (Y + B)(Y + X)$

[14] Note that the quiz provided to the students only included the instructions to "Solve the following for X:" and the 10 unsolved equations. We include the solutions and applied finance examples for each item to assist the reader in relating the original item to specific formulas used in introductory finance.

$$X = \frac{(Y + A)}{(Y + B)} - Y$$

Example: Find inflation given real and nominal interest rates. $[(1 + R) = (1 + r)(1 + h)]$

4) $Y = A + B(X - A)$

$$X = \frac{Y - A}{B} + A$$

Example: Find the expected market return given the expected return of the asset, risk free rate, and beta. $[E[R_i] = R_{rf} + \beta_i\left(E[R_m] - R_{rf}\right)]$

5) $Y = X(1 + A)^B$

$$X = \frac{Y}{(1 + A)^B}$$

Example: Find the initial investment that yields a specified future value given the rate of return and time. $[FV_t = PV_0(1 + r)^t]$

6) $Y = A(1 + X)^B$

$$X = \left(\frac{Y}{A}\right)^{1/B} - 1$$

Example: Find the rate needed to increase a given initial investment to a specified future value within a time period. $[FV_t = PV_0(1 + r)^t]$

7) $Y = A(1 + B)^X$

$$X = \frac{\ln\left(\frac{Y}{A}\right)}{\ln(1 + B)}$$

Example: Find the time needed to increase a given initial investment to a specified future value at a rate of return. $[FV_t = PV_0(1 + r)^t]$

8) $Y = \frac{1}{A}\sum_{i=1}^{A} B_i$ where A = 2, B_1 = A, B_2 = X

$$X = 2Y - 2$$

Example: Find the period-two return needed to yield an arithmetic mean two-period return given a period-one return. $\left[R_{arithmetic} = \frac{1}{N}\sum_{i=1}^{N} r_i\right]$

9) $Y = (\prod_{i=1}^{A}(1 + B_i))^{1/A} - 1$ where A = 2, B_1 = A, B_2 = X

$$X = \frac{(Y + 1)^2}{3} - 1$$

Example: Find the period-two return needed to yield a geometric mean two-period return given a period-one return. $\left[R_{geometric} = (\prod_{i=1}^{N}(1 + r_i))^{1/N} - 1\right]$

10) $Z = A^2X^2 + B^2Y^2 + AB$ where A = B

$$X = \sqrt{\frac{Z}{A^2} - Y^2 - 1}$$

Example: Find the standard deviation of returns for one asset given portfolio variance, variance of the other asset, and an equally weighted two-asset portfolio with covariance = 0.5. $\left[\sigma^2_{R_{2\_asset}} = x_1^2\sigma^2_{r_1} + x_2^2\sigma^2_{r_2} + 2x_1x_2\mathrm{Cov}(r_1, r_2)\right]$

APPENDIX B: Variable Definitions

Variable Name	Definition
Minority	Dummy variable equal to zero if student identified as "White", "Asian", or "International" and one if "Black or African American", "Hispanic", "Two or More Races", "American Indian or Alaska Native", "Native Hawaiian or Other Pacific Islander", or "No Response".
Female	Dummy variable equal to zero if student identified as male and one otherwise.
Other Major	Dummy variable equal to one if university records does not include a finance code for one of two possible majors at the time of enrollment in the course and zero otherwise.
Finance Minor	Dummy variable equal to one if university records include a finance code for one of three possible minors at the time of enrollment in the course and zero otherwise.
Age	Student age in years on date of the math pretest data collection. Data collection dates are 7SEP16 and 10JAN17.
Credits Earned	Number of completed credit hours as recorded by the registrar, including transfer credits.
Math ACT	Score on the mathematics portion of the ACT achievement test for college admissions.
GPA	University grade point average for all non-transfer credit hours. Any GPA equal to zero is assigned as a missing value.
Math Prereq	The numerical equivalent grade earned in the mathematics pre-requisite course.
Stat Prereq	The numerical equivalent grade earned in the statistics pre-requisite course.
Accounting Prereq	The numerical equivalent grade earned in the accounting pre-requisite course.
Final Exam	Percentage correct on the 50-question equally weighted final exam resulting in possible scores of 0 to 100%.
Correct Pretest $_n$	Correct Pretest is a count variable of correct answers to math pretest problems. The six-question measure includes problems 1-5 and 10 while the eight-question measure also includes 8 and 9.
Correct Applied $_n$	Correct Applied is a count variable of correct answers to application questions on the 50-question final exam. The six-question measure includes final exam problems 8, 11, 13, 25, 34, and 36 while the eight-question measure also includes problems 9 and 37.
Total Gain $_n$	The difference between correct answers on the applied and the pretest questions.
Adjusted Final $_n$	Adjusted Final Exam is the percentage of final exam correct answers with all application questions of math pretest skills removed.

Robust Analysis: An Investments Class Project on a Shoestring

Paul J. Haensly and Prakesh Pai[1]

ABSTRACT

We propose an investments class project to help students recognize ways to evaluate the robustness of an analytical result and understand the importance of performing such an evaluation. In the first part of the project, students derive basic results for naïve diversification. Then they apply four methods for evaluating the robustness of their initial conclusions: simple replication, an alternative market proxy in the single-index market model, an alternative asset pricing model (the Fama-French Three-factor Model), and results from another historical period.

Introduction

We propose a class project suitable for an introductory undergraduate or graduate course on investments. The primary objective is to help students recognize ways to evaluate the robustness of an analytical result and understand the importance of performing such an evaluation. We illustrate with an analysis of naïve diversification. At the end of the project, students should be able to do the following:

- Explain what robust analysis is intended to achieve and why it is important.
- Identify at least three general techniques for evaluating robustness.
- Evaluate the robustness of an empirical claim in finance.

The project assumes that students have a basic understanding of Excel, including how to navigate in an Excel workbook, copy and paste data, enter formulas in Excel, and set up tables.

In the first part of the project, students derive basic results for naïve diversification. The four main conclusions of this initial analysis are that:

(a) diversification reduces the risk that, by chance or mistake, an investor chooses a portfolio with a large amount of uncompensated risk that could have been diversified away;
(b) dispersion of the cross-sectional distribution of ***total*** portfolio risk is small but not zero even at large portfolio sizes;
(c) dispersion of the cross-sectional distribution of ***diversifiable*** portfolio risk is not negligible even at large portfolio sizes; and
(d) shocks due to diversifiable risk in an N-stock portfolio are significant, even for large values of N.

These points have been demonstrated in the recent literature, *e.g.,* Bennett and Sias (2011). The last three conclusions are important because they contradict the advice based on early papers on naïve diversification in which the authors concluded that an investor needs only a small number of securities to be well diversified. This conclusion often is cited in textbooks and the popular press and is treated as a justification for active portfolio management. Thus, a secondary objective of this project is to illustrate the importance of critical reading skills. The proposed project helps students to recognize that published claims, even those in a textbook or scholarly paper, should be evaluated carefully.

A robust analysis is one in which the results continue to hold under a variety of conditions. The better the robustness of a study about investments, the more confident we are that the conclusions are a good guide for investment decisions. In the project, students examine four common methods for evaluating the robustness of an investment analysis of naïve diversification. The first is simple replication. If we repeat our sampling of portfolios at each portfolio size, do we get the same results as before? The second is the choice of an

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alternative market proxy in the single-index market model. Are the results sensitive to choice of a market index? The third is choice of an alternative asset pricing model. In this project, we compare results when using the single-index market model to those when using the Fama-French Three-factor Model (Fama and French 1992, 1993). The fourth is examination of results for another historical period. We illustrate by comparing an initial analysis on data for 2012-2016, which included a major bull market, with an analysis for 2007-2011, which included a stock market crash.

Why should a reader of this journal be interested? For teachers of investments courses, the proposed project may raise students' awareness of the risk of taking claims at face value. Introductory finance textbooks that present the basic results of naïve diversification do so in order to motivate an in-depth study of modern portfolio theory, but then undercut that justification by repeating the dogma that an investor needs only a small number of securities to be well diversified. If students first read the textbook and then participate in our project, they may be more motivated to ask questions and gain a deeper understanding of later material in the course. For teachers of other courses in economics and finance, our proposed investments class project may serve as insight and motivation for robustness analysis in their courses.

We organize the paper as follows. First, we briefly review the literature on naïve diversification. Then we describe the data for the project. Next, we outline the baseline analysis and summarize the main conclusions. Finally, we describe the robustness tests and ways to present the results. We conclude with a summary of the robustness analysis and lessons that we hope students learn.

Review of the Literature

Early research on naïve diversification focused on the asymptotic decline of average total risk. See, *e.g.*, Evans and Archer (1968), Johnson and Shannon (1974), Bird and Tippett (1986), and Statman (1987). These authors show that diversification by means of adding stocks to an equal-weighted portfolio decreases total risk. Moreover, the marginal reduction in average total risk declines and rapidly approaches the level of market risk. The authors typically conclude that eight to 20 stocks are sufficient to reduce total risk to the level of market risk.

More recent papers evaluate the cross-sectional ***dispersion*** of portfolio risk, conditional on the number of securities in the portfolio. See, *e.g.*, Newbould and Poon (1996), Surz and Price (2000), and Bennett and Sias (2011). These authors show that conclusions based only on the average total risk are misleading. The range of cross-sectional total risk is nonzero even for portfolio sizes as large as 300 stocks. These results suggest that far more than 20 stocks are needed to assure adequate diversification. Unfortunately, finance textbooks tend to focus on the early conclusions in the literature, but not the latter. Tang (2004) finds that authors of investment and financial management textbooks conclude that eight to 40 stocks are sufficient to eliminate most diversifiable risk in a portfolio.

Data

This project draws on readily available, free sources of data on the Internet. Hence, faculty at small universities who have little or no funding for financial databases can carry out this project. The instructor collects and organizes the data in Excel workbooks set up to facilitate analysis by the class. Here, we briefly describe the data, which consists of individual stock prices and market index returns.

The individual stock prices come from Yahoo! Finance (https://finance.yahoo.com), which provides daily closing prices adjusted for cash dividends and stock splits. Hence, we can calculate total returns directly from these adjusted prices. For this paper, we draw the list of stocks from the portfolio composition file (PCF) for the Vanguard Large Cap Exchange Traded Fund (ETF) (Vanguard 2017). (The URL in the References links to the current PCF file for this ETF. To navigate to the PCF files for other Vanguard ETFs, go to Vanguard's ETF page [https://investor.vanguard.com/etf/], navigate to the main page of the desired ETF, click on the "Portfolio & Management" tab, and locate the link "View the PCF for this Vanguard ETF.")

The Vanguard Large Cap ETF tracks the CRSP U.S. Large Cap Index. We chose it because the stocks in this index constitute about 85% of the market capitalization of the U.S. stock market. In principle, the project can be based on any reasonably comprehensive list of stocks. However, a list derived from a broad-based ETF of U.S. stocks provides some assurance that a reasonably large proportion of the list will have usable data in Yahoo! Finance. From the 602 stocks in this ETF as of May 15, 2017, we illustrate this project with the 498 stocks that have complete time series of daily price data over the ten-year target period, 2007-2016.

(A list of these stocks is available from the authors.) The instructor converts the daily adjusted closing prices for each stock into monthly total returns. Then, using the one-month risk free rate available at the Kenneth R. French Data Library (2018), the instructor constructs time series of excess monthly total returns for 2007-2016.

The Kenneth R. French Data Library (2018) provides excess monthly total returns on the U.S. stock market. The total market index from this data library is a value-weighted index formed from all CRSP firms incorporated in the U.S. and listed on the major U.S. stock markets. French constructs this index from high-quality CRSP data and calculates returns in excess of the one-month Treasury bill rate. French also provides monthly returns on two additional benchmark factors needed to apply the Fama-French Three-factor Model. Please see Fama and French (1993) for a complete description of these factors.

Baseline Analysis

In this section, we describe the initial assessment of naïve diversification. The class performs an empirical analysis of the cross-sectional distributions of total portfolio risk, systematic risk, and diversifiable risk. The instructor provides template Excel workbooks for carrying out the sampling and computations. (The interested reader may contact the authors for an example template workbook.) The instructor divides the data analysis tasks among the students. Once completed, the instructor collates results from the students and distributes the data on the cross-sectional distributions for further analysis.

The class analyzes risk for equal-weighted portfolios rebalanced monthly. To keep the project manageable, we recommend that the instructor choose a representative selection of portfolio sizes (*i.e.*, number of stocks) as illustrated here. At each portfolio size greater than N = 1, the class draws a sample of 1,000 portfolios of N stocks each. (For N = 1, the class simply uses the population of 498 stocks for one-stock portfolios.) The sampling proceeds in two stages. To draw the list of stocks for a given portfolio, students perform simple random sampling without replacement from the list of stocks. They repeat this sampling to create the sample of portfolios (out of the population of all possible portfolios of size N).

The instructor sets up the template Excel workbook for each portfolio size N so that the student only needs to create a table of random numbers and paste it into the workbook. We set up our template Excel workbook so that the student enters a table of 498 by 100 random numbers. The workbook takes each column of 498 random numbers to perform the sampling required to identify N stocks for each of 100 portfolios. Once a portfolio has been identified, the workbook calculates the excess monthly total returns and then the risk statistics. The instructor collates the separate student-generated samples of 100 portfolios of size N to form a combined sample of 1,000 portfolios.

Choice of portfolio sizes N to examine is not critical. Because the cross-sectional medians (and other percentiles) of the risk statistics tend to level off fairly quickly as N increases, it generally is sufficient in the baseline analysis to use the same values of N as, for example, in Table 4.8 in Elton et al. (2010).

Yahoo! Finance does not provide historical data about shares outstanding, float, or market capitalization other than for the most recent quarter. Hence, we are limited to working with equal-weighted portfolios. The instructor can point out to the class that most papers on naïve diversification assume equal-weighted portfolios. Thus, the project results are directly comparable to the claims that we are critiquing.

As part of the data analysis of the risk statistics, the instructor can direct the class to apply the SKEW function in Excel to evaluate symmetry of the cross-sectional distributions of total portfolio risk at each size N. This analysis shows students that these distributions are skewed, especially at smaller portfolio sizes. Hence, means and standard deviations are not the best choice for describing location and dispersion. This result justifies the application of nonparametric statistics to describe the cross-sectional distributions: the median for location, and selected percentile statistics to evaluate dispersion.

To partition total portfolio risk for each portfolio into its systematic and diversifiable components, we apply the single index market model (SIMM) under the standard assumptions about covariances between error terms and the market index and between error terms for different stocks:

$$\tilde{r}_{it} = \alpha_i + \beta_{mi}\tilde{r}_{mt} + \tilde{e}_{it}, \tag{1}$$

where $\tilde{r}_{it}$ is the total return on stock i in period t, in excess of the one-month T-bill return, with standard deviation σ_i; $\tilde{r}_{mt}$ is the total return on the market index m in period t, in excess of the one-month T-bill return, with standard deviation σ_m; and $\tilde{e}_{it}$ is the error term for stock i in period t, with mean zero and finite standard deviation σ_{ei}. For equal-weighted portfolios rebalanced each period, the SIMM is

$$\tilde{r}_{pt} = \sum_{i=1}^{N} w_{it}\tilde{r}_{it} = \sum_{i=1}^{N} \frac{1}{N}(\alpha_i + \beta_{mi}\tilde{r}_{mt} + \tilde{e}_{it}) = \alpha_p + \beta_{mp}\tilde{r}_{mt} + \tilde{e}_{pt}, \tag{2}$$

where $\tilde{r}_{pt}$ is the excess portfolio total return in period t, α_p is an average of the alphas, β_{mp} is an average of the betas, and $\tilde{r}_{mt}$ is excess total return on the market index. The partition of total risk for portfolio p takes the form

$$\sigma_p^2 = \beta_{mp}^2\sigma_m^2 + \sigma_{ep}^2, \tag{3}$$

where the first term on the right-hand side is systematic risk, the second term is diversifiable risk, and

$$\sigma_{ep}^2 = var(\tilde{e}_{pt}) = \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N} \sigma_{ei}^2. \tag{4}$$

An attractive feature of this partition is that it is additive. However, the scale is in terms of returns squared. Hence, when interpreting and analyzing these components, it is more helpful to work with systematic risk expressed as $\beta_{mp}\sigma_m$ and diversifiable risk expressed as σ_{ep}. In our template workbook, we generate the monthly excess portfolio returns and then estimate systematic and diversifiable risk for each portfolio by applying least squares regression directly to its time series of returns. This approach is more computationally efficient in Excel than first estimating beta and the error term variance for individual stocks and then using these results to estimate portfolio beta and σ_{ep}^2.

Because the instructor sets up the Excel workbook, there is a danger that it becomes a black box. This problem is inherent with prepackaged software in general. Hence, we recommend that the instructor require students to take at least one of their portfolios and carry out all of the calculations themselves. Students thus gain an understanding of what the "black box" is doing. In addition, if they set up their calculations in a separate Excel worksheet, students can check a few of their portfolios to verify that the Excel workbook is producing the correct results. While this task is not a robustness check, it does help students recognize that we should not take output from software packages for granted.

In the baseline analysis, the class evaluates the cross-sectional distributions of risk based on monthly returns for 2012-2016. Students construct plots of the median of the estimated total portfolio risk $\hat{\sigma}_p$ (the estimated monthly standard deviation of portfolio excess return) and selected percentiles for describing the dispersion of $\hat{\sigma}_p$. Figures 1A and 1B illustrate charts that we recommend the instructor ask students to construct. These charts show that total portfolio risk falls, on average, as number of stocks in portfolio increases, and the dispersion of cross-sectional total risk decreases. For the data that we use to illustrate this project, most portfolios of 300 stocks (the largest portfolio size that can be handled conveniently in Excel) have monthly total risk within about 0.2% of the market risk; see Figure 1B. The 80th percentile for $\hat{\sigma}_p$ is 3.26% per month versus market risk of 3.12% per month. This result is consistent with our intuition that a sufficiently diversified portfolio has total risk that is about the same as its market risk.

The instructor can point out that, up to the point where students plot the *median* total risk in Figures 1A and 1B, they have replicated the chart presented in the typical textbook, but now we start looking more closely at the results. For starters, Figure 1B is helpful, because students can see that the scale in Figure 1A is driven by the most volatile portfolios (*i.e.*, one-stock portfolios). Otherwise, Figure 1A by itself appears to lend support to the notion that diversifying beyond 30 stocks does not offer much benefit. In addition, the instructor should now call attention to the *dispersion* of risk (which usually is not included in the typical textbook chart). For portfolios of 30 stocks, a sizeable minority of portfolios has monthly total risk that differs from the market risk by 0.5% or more. The 80th percentile for $\hat{\sigma}_p$ is 3.63% per month versus market risk of 3.12% per month. (For perspective, it may help if the instructor asks students to annualize these values, which are approximately 12.57% and 10.81%, respectively, when annualized.)

Figure 1A. Cross-sectional Distribution of Total Portfolio Risk, N ≤ 50 (partition based on 1-factor market model; 2012-2016 data)

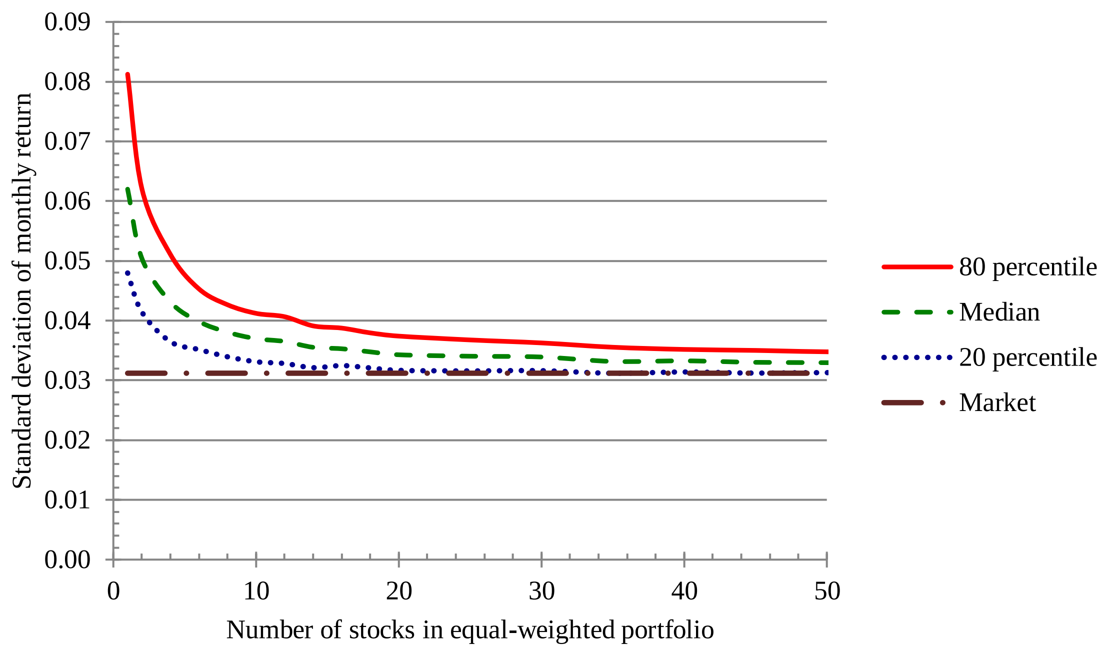
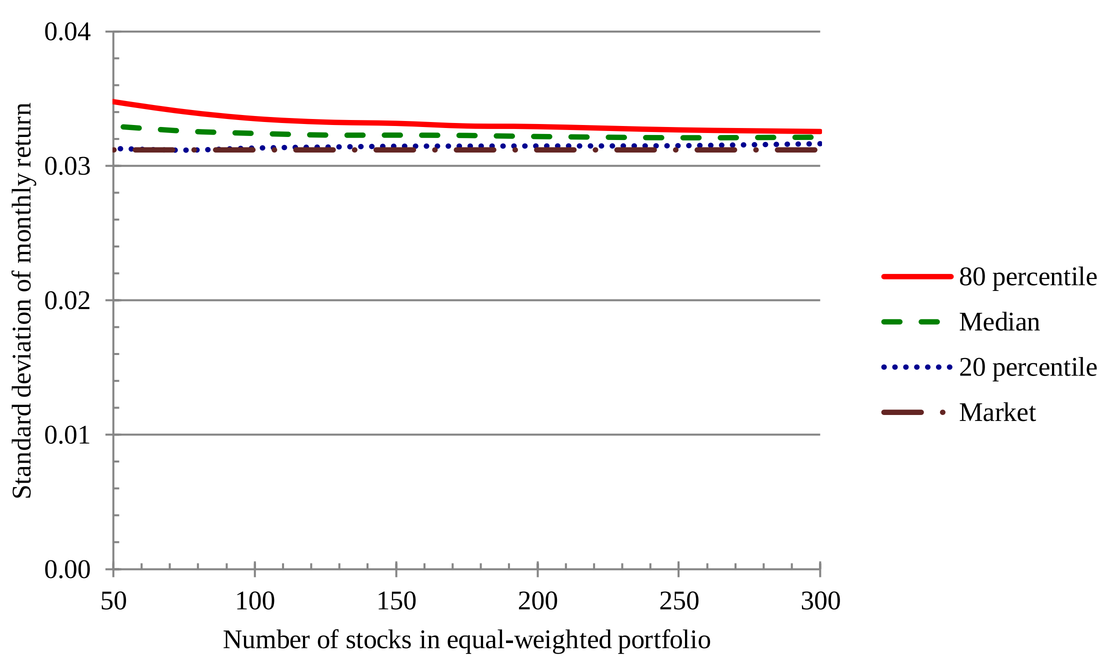


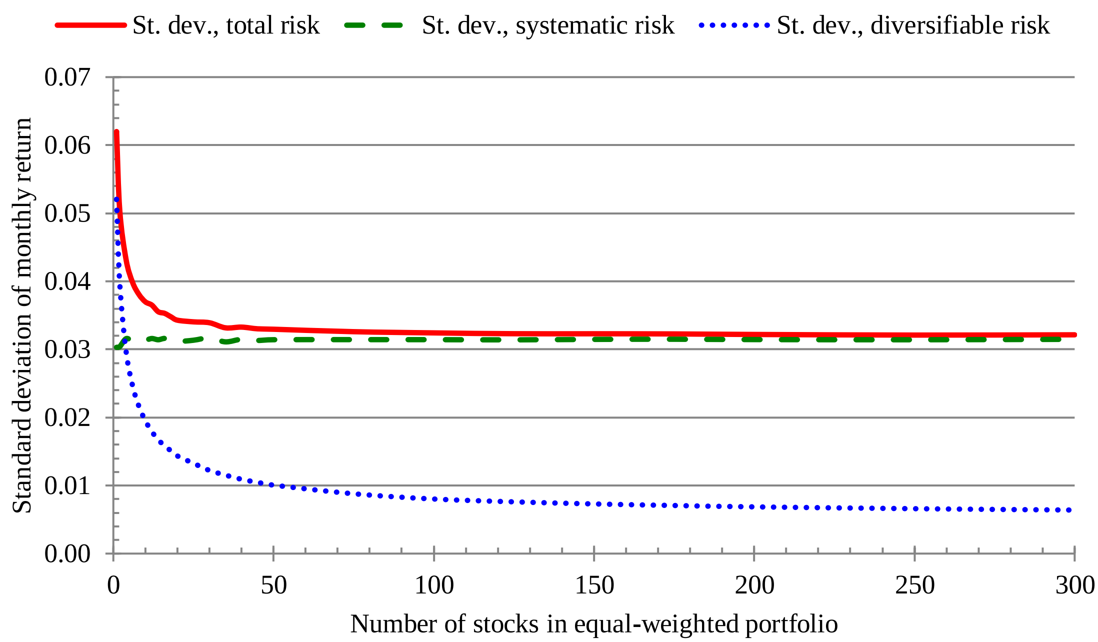
Figure 1B. Cross-sectional Distribution of Total Portfolio Risk , N ≥ 50 (partition based on 1-factor market model; 2012-2016 data)



A generally accepted principle in finance is that investors are not compensated for risk that they can easily diversify away. Diversification should reduce the likelihood that, ***by chance or mistake***, an investor chooses a portfolio with a large amount of diversifiable risk. While Figures 1A and 1B show that total risk declines on average as number of stocks in the portfolio increases, it is important that students recognize that we have not yet demonstrated directly that diversifiable risk has been reduced.

To examine this question, the instructor directs students to construct a chart, such as Figure 2, showing the medians of the cross-sectional distributions of monthly total risk, systematic risk, and diversifiable risk. This chart illustrates two important points. First, when we work on the same scale as rate of return, the portfolio decomposition is not additive as it is in Equation (3). Second, while the median total risk converges quickly to the median systematic risk, ***the median diversifiable risk does not converge quickly to zero***. Even for portfolios of 300 stocks, the estimated median diversifiable risk is 0.64% (or 2.22% when annualized). Discussion about the effectiveness of diversification often is framed in terms of how close total risk is to systematic risk. An important conclusion that students should draw from Figure 2 is that diversifiable risk is ***not*** correctly estimated as the difference between total and systematic risk.

Figure 2. Medians: Total, Systematic, and Diversifiable Risks
(decomposition based on 1-factor market model; 2012-2016 data)

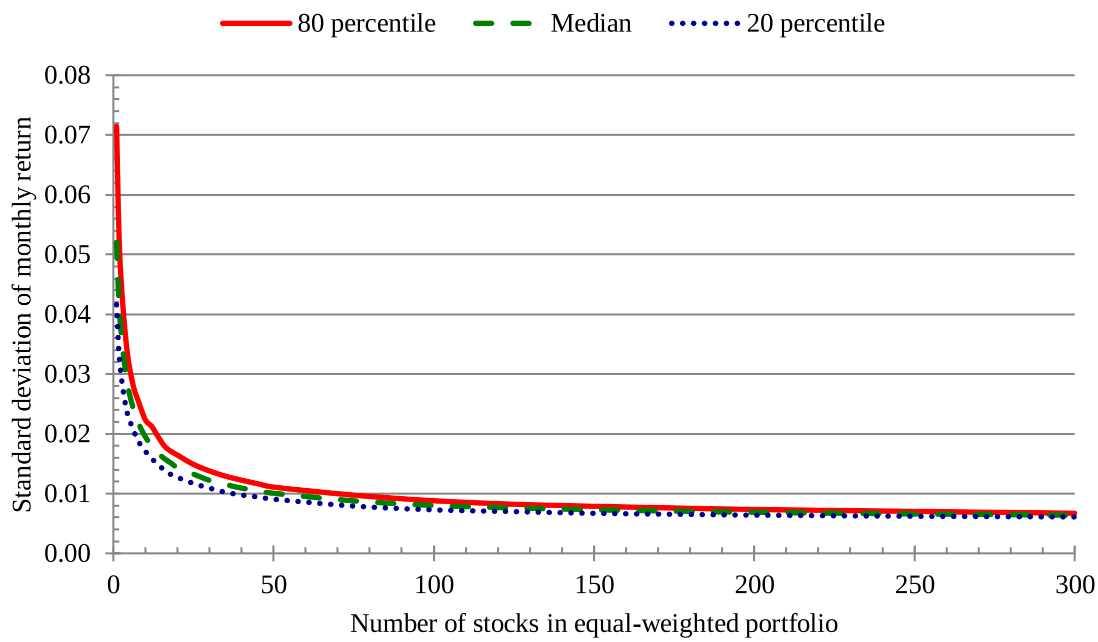


This result leads us to focus on the diversifiable risk. In the baseline analysis, the instructor directs students to construct charts showing the cross-sectional distributions of the monthly diversifiable risk at each portfolio size. As expected, it falls, on average, as number of stocks in the portfolio increases, and dispersion also decreases. What is not expected is that the level of diversifiable risk does not vanish even for 300-stock portfolios. Figure 3 illustrates these results. For 80 percent of portfolios at this size, the monthly diversifiable risk is above the 20th percentile of 0.61% (above 2.11% when annualized), given the data in this paper.

This result may puzzle students. How can total risk converge to market risk, yet significant diversifiable risk remains? Helping students to unravel this mystery also helps to explain why many scholars and professionals in finance have been tripped up by the same question. Recall that the partition of total risk defined by Equation (3) is, indeed, additive. To visualize this additivity, students construct a chart, such as Figure 4, that displays the median of the empirical cross-sectional distributions of total risk, systematic risk, and diversifiable risk in terms of ***variance*** of return.

The additivity in Equation (3) applies to each portfolio. However, medians of each type of risk are not necessarily additive, because they likely are for different portfolios. Nonetheless, they are approximately additive for all except the one-stock portfolios. In Figure 4, students observe two intuitively appealing results: the median total portfolio risk converges (rapidly) to the median systematic risk, and the median diversifiable risk converges (rapidly) to zero.

Figure 3. Dispersion of Monthly Diversifiable Risk
(partition based on 1-factor market model; 2012-2016 data)



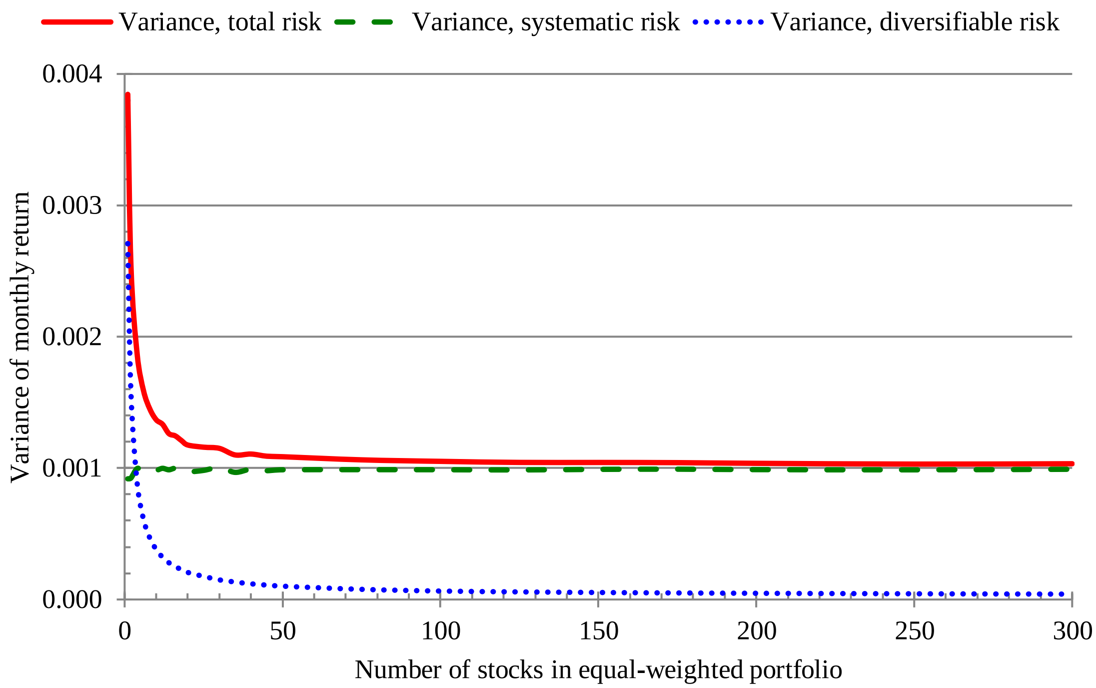
An interesting exercise is to see if students can reconcile the results in Figures 2 and 4. The explanation is that the square root function, $\sqrt{x}$, is concave with extremely steep slope close to zero but flattening out as x increases. Thus, converting from variance to standard deviation shifts up the very small values of variance of diversifiable risk much more than it shifts up the relatively larger values of variance of systematic and total risk. To illustrate, consider the following medians for N = 30 for the data in this paper: the estimate of variance of total risk, 0.001150; the estimate of variance of systematic risk, 0.001004; but the estimate of variance of diversifiable risk, 0.000149, is an order of magnitude smaller than either total risk or systematic risk. Now consider the medians in terms of standard deviation of risk: total risk, 0.033912; systematic risk, 0.031686; and diversifiable risk, 0.012207. After taking the square root, the diversifiable risk now is roughly one third of the values of total or systematic risk.

The initial stage of the project wraps up with a class analysis in which the instructor guides students in an effort to quantify the shocks due to diversifiable risk and assess the implications for an investor. These shocks arise from portfolio error terms, $\tilde{e}_{pt}$, in the SIMM where, by assumption, the means are zero.

For the stocks in this paper, the median diversifiable risk for 30-stock portfolios is 1.22% per month. The instructor should ask students to first consider a hypothetical portfolio with this level of diversifiable risk and apply basic probability rules to interpret it. For example, under the assumption that the error terms follow a normal probability distribution, there is about a 16 percent chance that the monthly shock due to diversifiable risk in this portfolio will be -1.22% or worse, or about -4.23% when annualized (assuming that monthly error terms are independent). Based on our reasoning for this hypothetical portfolio, the instructor then guides students to recognize that if a portfolio has diversifiable risk greater than the median of 1.22%, then the probability must be ***greater*** than 16 percent that the portfolio will have a shock of -1.22% or worse per month. But this conclusion then must apply ***to half of all possible 30-stock portfolios***.

For 300-stock portfolios, the median diversifiable risk is 0.64% per month. Under the same distributional assumptions for the portfolio error terms, there is about a 16 percent chance that the monthly diversifiable shock will be -0.64% or worse, or an annualized shock of about -2.21%. These shocks illustrate that significant diversifiable risk remains for a large proportion of portfolios even when the number of stocks in the portfolio is large.

Figure 4. Partition of Total Risk In Terms of Variance of Portfolio Return: Medians (partition based on 1-factor market model; 2012-2016 data)



Robustness of the Results

In the next stage of the project, the instructor guides students in the evaluation of the robustness of the baseline analysis of naïve diversification. We propose that the class address four questions.

(a) Are the results robust to replication with new random samples?
(b) Are the results robust to plausible alternatives for the market proxy, since the choice of a market index plays a crucial role in the risk decomposition?
(c) Are the results robust to choice of a model, since the asset pricing model defines the decomposition of total risk?
(d) Are the results robust to analysis for alternative historical periods, since the results may be unique to the specific historical period for the returns data?

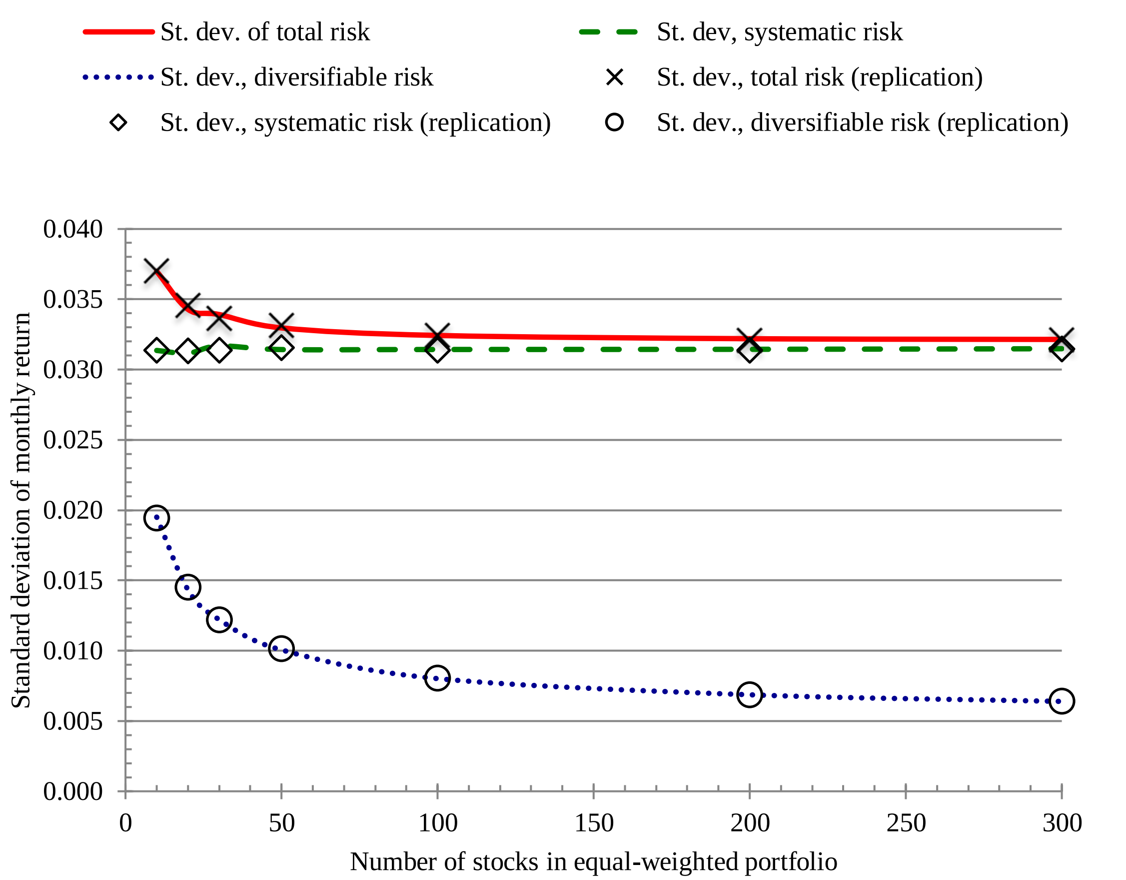
In a classroom project, it is important not to overwhelm students with work that can be tedious and thus cause them to lose sight of the basic principles. Hence, for this project, the class reexamines the original results for a limited but representative subset of portfolio sizes examined earlier. We illustrate with N = 10, 20, 30, 50, 100, 200, and 300 stocks. This choice avoids the scale problem mentioned earlier, yet covers a sufficient number of portfolio sizes to be representative of the trends as N increases.

Replication

Random sampling introduces variability in statistics simply due to random chance. In the analysis of naïve diversification, the instructor asks students to think about the sample size compared to the population of possible portfolios. This task is a basic combinatorial problem that (we hope) students recall from their statistics course. If we draw simple random samples without replacement of N stocks from a population of *m* stocks, how many distinct portfolios can we construct? For example, if N = 10 and *m* = 498, then $\binom{498}{10} = 2.36 \times 10^{20}$, which students can calculate using the COMBIN function in Excel. If we draw 1,000 portfolios, then we selected only a tiny fraction of all possible portfolios. Thus, an important question is how representative a given sample of 1,000 portfolios is of the population. Equivalently, we want to know how **resistant** our measures of location (*e.g.*, the median) and dispersion (*e.g.*, pairs of percentile statistics, such as the 20th percentile and 80th percentile) are to drawing a new sample of 1,000 portfolios.

From the baseline analysis, each student has a collection of specially constructed Excel workbooks in which the student loaded a table of random numbers. Each workbook then carries out the sampling for 100 portfolios of size N and computes the risk statistics. To perform the replication, each student creates copies of his or her original workbooks, loads a new table of random numbers, and then sends the results to the instructor to collate and then distribute for further analysis. For example, the instructor can direct the class to construct a chart in which they plot the medians for each type of risk (total, systematic, and diversifiable) at each portfolio size. In the chart, the student first plots the original results and then overlays the results from the replication. See Figure 5 for an illustration.

Figure 5. Medians: Total, Systematic, and Diversifiable Risks
(decomposition based on 1-factor market model; 2012-2016 data)



For sample sizes of 1,000 portfolios of large-cap stocks, Figure 5 shows that the median, 20th percentile, and 80th percentile statistics change very little from one sample to the next. Students should be able to draw the statistical conclusion that these measures of location and dispersion are resistant for this sample size (1,000 portfolios) and for the portfolio sizes in the replication analysis (10-stock to 300-stock portfolios). Thus, the empirical cross-sectional distributions of a given sample of 1,000 portfolios do not change much from one sample to the next, *i.e.*, our original results are robust with respect to the random sampling.

An important question is what the above results imply for the remaining analysis of robustness. The instructor explains that, since our risk measures are resistant, a given sample of portfolios is sufficiently representative that we can apply matched pairs comparisons (and, if the class is up to formal statistical analysis, matched pairs hypothesis tests). In a matched pairs analysis, two variables are measured for each portfolio in the sample, and we evaluate the difference.

Choice of Market Proxy in the Single Index Market Model

The market index selected for the SIMM plays an important role in determining the partition described in Equation (3). The estimators for systematic and diversifiable risk are determined by choice of the market index. The instructor reminds the class that we estimate systematic risk by $\hat{\beta}_{mp}\hat{\sigma}_m$, where $\hat{\beta}_{mp}$ is the least-squares estimate of portfolio beta for the regression of the excess total portfolio return on the excess market return and $\hat{\sigma}_m$ is the square root of the sample estimate of variance of the excess market return. We can estimate diversifiable risk by the square root of the residual mean square for the regression of portfolio return on market return.

An alternative market index that students can easily construct with the data at hand is the excess return on an equal-weighted portfolio (rebalanced monthly) of all of the available stocks in the study (for the data in this paper used to illustrate the project, the 498 stocks identified in the Data section). This market proxy is the most common choice in the literature on naïve diversification. By rerunning the analysis with this market index and comparing the results to the baseline case, the class can not only can evaluate robustness but also replicate the type of analysis common in the literature.

The first step is for students to construct the excess total return on an equal-weighted portfolio of the 498 stocks. This task is easily accomplished in Excel using a table of the monthly returns on the individual stocks and the monthly risk-free rate. Then students substitute this new market index in the same cells as the total stock market index in their Excel workbooks used in the baseline analysis. The next step is to generate the risk statistics for each portfolio, conditional on portfolio size. The Excel workbooks perform this task automatically. Students then report the results for their sets of portfolios to the instructor, who collates them and returns the statistics to the class for further analysis.

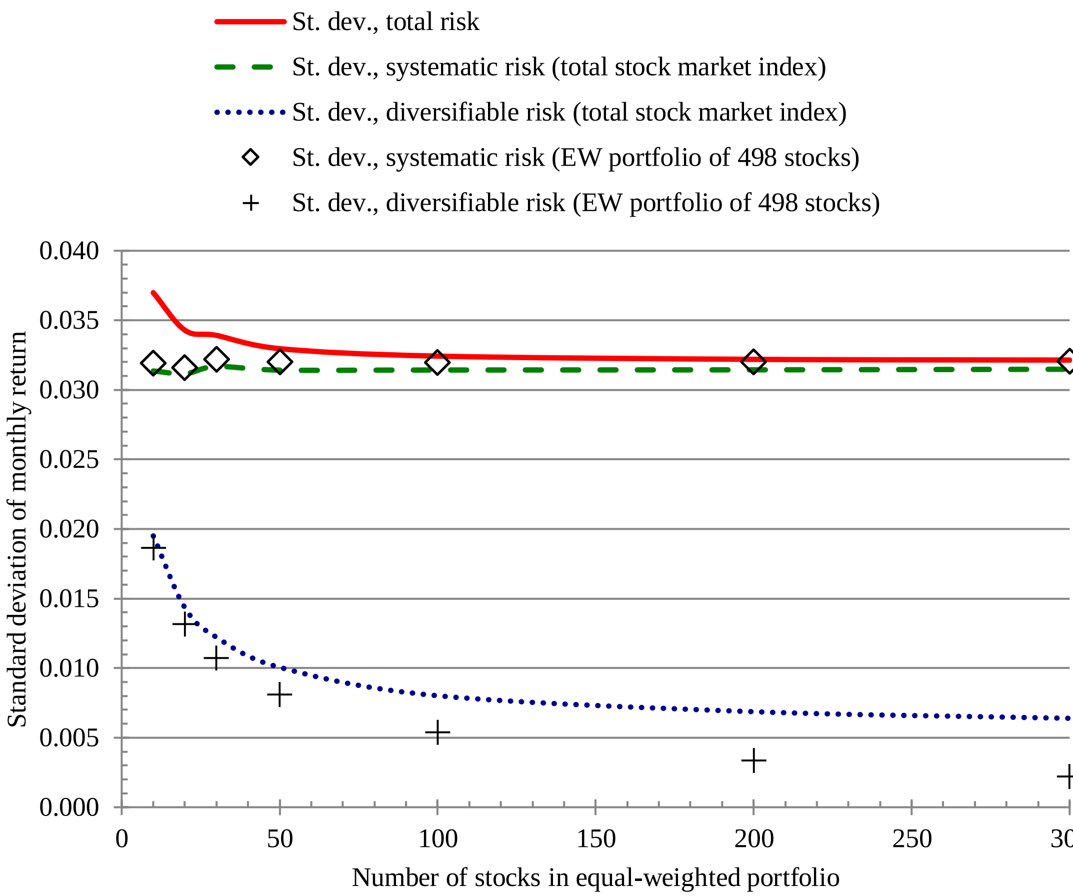
The instructor directs students to construct a chart that displays the medians of total risk, systematic risk, and diversifiable risk from the baseline study (in which the market index is the total U.S. stock market index), then overlay the median statistics for systematic and diversifiable risk when the market index is the equal-weighted (EW) index of the stocks in the study. See Figure 6 for an illustration. (In a matched pairs analysis, we use the same portfolios in both cases. Hence, the standard deviations of total risk are the same regardless of the market index.)

The general pattern is the same as in Figure 4 where the market index is the excess CRSP Total U.S. Stock Market Index. The median total risk declines quickly and asymptotically approaches the median portfolio market risk; diversifiable risk declines at a decreasing rate as N increases but is not zero for N as large as 300 stocks. An important difference is that the partition with the new market index shifts more of the total risk to systematic risk. In Figure 6, we see that the median portfolio market risk with the new index is consistently higher than before, and the median diversifiable risk becomes noticeably lower as N increases.

The class should discuss why this result makes sense. Specifically, students should observe that their equal-weighted portfolios more quickly become like the "market" as N increases when the "market" (a) follows the same weighting scheme as the portfolios of size N (equal weighted rather than value weighted), (b) is much smaller (in this case, 498 stocks versus several thousand in the CRSP Total U.S. Stock Market Index), and (c) consists of the same types of stocks (large caps rather than a range of market capitalization).

The instructor might also present two other points for the class to consider. First, what does the result with the equal-weighted market index constructed from the stocks at hand (the 498 stocks in our running example) tell us about diversification in practice? Figure 6 suggests that if the portfolio really does get close to the market in terms of number of stocks, then diversifiable risk might really be reduced to negligible levels. However, if the market consists of a very large number of securities, then the well-diversified portfolio will also need to include a substantial proportion of these securities. Second, what does the result tell us about the conclusions in previous studies on naïve diversification? Figure 6 suggests that these papers overstate the effectiveness of diversification when number of securities in the market proxy is much less than in the total stock market.

Figure 6. Medians: Total, Systematic, and Diversifiable Risks (1-factor market model with different market proxies; 2012-2016 data)



The final comparison concerns shocks due to diversifiable risk in the portfolio (*i.e.*, that due to company-unique risks). The instructor again needs to provide careful guidance about the interpretation of the statistics. Suppose that the diversifiable risk for a given portfolio is q% per month. Under the assumption that the error terms follow a normal probability distribution, there is about a 16 percent chance that the monthly shock due to diversifiable risk will be $-q\%$ or worse, or about $-q\sqrt{12}\%$ when annualized (assuming that monthly error terms are independent). Table 1 illustrates how the class might make comparisons for portfolios with the median diversifiable risk, and we can perform an analogous comparison for other percentiles of the diversifiable risk.

A critical point that may take students time to digest is that Table 1 is for portfolios with the median shock due to diversifiable risk at each size. Hence, this table ***represents the best case for half of the portfolios at each size***. We can express this point in two ways. For example, for 30-stock portfolios, the table lists an annualized shock of -4.23% for a portfolio with the median diversifiable risk. Hence, for half of all 30-stock portfolios, there is a 16 percent chance that the annualized shock will be ***worse*** than -4.23%. Alternatively, for half of all 30-stock portfolios, the chance of an annualized shock of -4.23% or worse is ***greater*** than 16 percent.

These results show that while the general pattern of risk as N increases is the same for both proxies for the market, the estimates of diversifiable risk, especially for larger values of N, are not robust. Switching from a value-weighted total market index to an equal-weighted index composed only of stocks in the study leads to lower estimates of portfolio diversifiable risk and hence lower estimates of shocks due to this risk. This result is consistent with one of the concerns raised by Roll (1977). Namely, even if market indexes are highly correlated, the use of different market proxies can lead to significantly different conclusions.

Table 1. Unsystematic Shocks Due to Diversifiable Risk Remaining in the Portfolio: Upper End of Range of Shocks That Have an Approximately 16% Chance of Occurring for Portfolios With the Median Diversifiable Risk; Equal-weighted Large Cap Portfolios, 2012-2016

	Market proxy is CRSP Total U.S. Stock Market Index		Market proxy is equal-weighted portfolio of the 498 stocks in the study	
Size of portfolio	Monthly unsystematic shock	Annualized unsystematic shock	Monthly unsystematic shock	Annualized unsystematic shock
10	-1.95%	-6.76%	-1.86%	-6.46%
20	-1.43%	-4.97%	-1.32%	-4.56%
30	-1.22%	-4.23%	-1.07%	-3.72%
50	-1.01%	-3.48%	-0.81%	-2.80%
100	-0.80%	-2.78%	-0.54%	-1.87%
200	-0.69%	-2.38%	-0.34%	-1.16%
300	-0.64%	-2.21%	-0.22%	-0.77%

Notes. The unsystematic shocks in this table assume a single index market model with error terms that have a normal probability distribution with mean of zero. Size of portfolio is the number of stocks in an equal-weighted portfolio rebalanced monthly.

Choice of Asset Pricing Model

In the next part of the robustness analysis, the class compares the risk decompositions when using the SIMM to that when using the Fama and French Three-factor Model (FF3M); see Fama and French (1992, 1993). We can write this model as

$$\tilde{r}_{pt} = \alpha_p + \beta_{mp}\tilde{r}_{mt} + \beta_{SMB,p}\tilde{r}_{SMB,t} + \beta_{HML,p}\tilde{r}_{HML,t} + \tilde{e}_{pt}, \tag{5}$$

where $\tilde{r}_{pt}$ is the total return on portfolio p in period t, in excess of the one-month T-bill return, with standard deviation σ_p; $\tilde{r}_{mt}$ is the total return on the market index m in period t, in excess of the one-month T-bill return, with standard deviation σ_m; $\tilde{r}_{SMB,t}$ is the return on the Fama and French SMB (small minus big) portfolio, which serves as the risk factor related to firm size, with standard deviation σ_{SMB}; $\tilde{r}_{HML,t}$ is the return on the Fama and French HML (high minus low) portfolio, which serves as the risk factor related to book-to-market equity, with standard deviation σ_{HML}; and $\tilde{e}_{pt}$ is the error term for portfolio p in period t, with mean zero and finite standard deviation σ_{ep}.

Under standard assumptions for the above model, we can show that the partition of total risk for portfolio p is

$$\sigma_p^2 = \beta_{mp}^2\sigma_m^2 + \beta_{SMB,p}^2\sigma_{SMB}^2 + \beta_{HML,p}^2\sigma_{HML}^2$$

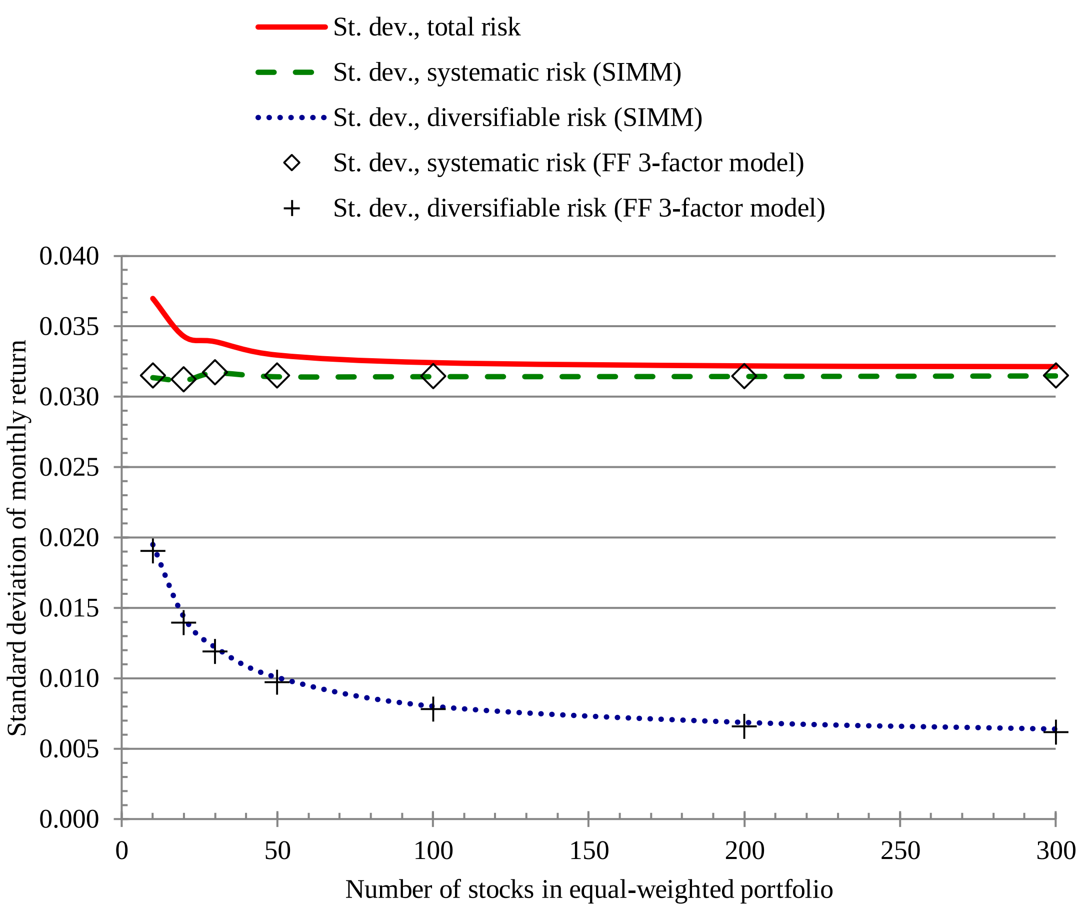
$$+2\beta_{mp}\beta_{SMB,p}\sigma_{m,SMB} + 2\beta_{mp}\beta_{HML,p}\sigma_{m,HML} + 2\beta_{SMB,p}\beta_{HML,p}\sigma_{SMB,HML} + \sigma_{ep}^2, \tag{6}$$

where $\sigma_{m,SMB} = cov(\tilde{r}_{mt}, \tilde{r}_{SMB,t})$, $\sigma_{m,HML} = cov(\tilde{r}_{mt}, \tilde{r}_{HML,t})$, and $\sigma_{SMB,HML} = cov(\tilde{r}_{SMB,t}, \tilde{r}_{HML,t})$. The sum of all terms on the right-hand side except the last term is the **systematic risk** of portfolio p. The last term, σ_{ep}^2, is the **diversifiable risk**. Factor models often are constructed so that the factors are uncorrelated. This feature makes it easier to distinguish the importance of the different factors. For the study of naïve diversification, however, the assumption that the factors are uncorrelated is unnecessary. We simply wish to partition total risk into systematic risk and diversifiable risk. Moreover, the factor correlations are sufficiently low so as to avoid the computational problems that arise when factors are highly correlated.

As with analysis using the SIMM, the instructor sets up a template Excel workbook that carries out the computations of the estimates of systematic and diversifiable risk for each portfolio once the student has selected the stocks by simple random sampling without replacement. Students use the workbooks to generate the risk statistics for subsets of portfolios at each size N. We set up the template Excel workbook to simultaneously carry out the estimation for the SIMM and the FF3M. Thus, the risk estimates are already available for the latter model if we use the same portfolios as in the baseline analysis (and thus perform a matched pairs analysis). In that case, differences in results are due solely to choice of the asset pricing model. The instructor collates the results and returns the risk statistics to the class for analysis.

The instructor directs students to construct a chart comparing median systematic and diversifiable risks for the two asset pricing models across the portfolio sizes. See Figure 7 for an illustration. For these portfolio sizes, there is virtually no difference in each type of risk at each size.

Figure 7. Medians: Total, Systematic, and Diversifiable Risk (1-factor market model vs. Fama-French 3-factor model; 2012-2016 data)



The instructor then asks students to compare dispersion of systematic and diversifiable risks when we use the SIMM versus when we apply the FF3M. A chart is visually helpful in seeing that the dispersion (*e.g.*, as measured by the difference between the 80th and 20th percentiles) is virtually the same, given the portfolio size. (We do not include an example of this chart.) In summary, based on these observations about location and dispersion of the cross-sectional distributions of diversifiable risk, the estimates of shocks due to diversifiable risk will be approximately the same regardless of the choice of asset pricing model.

Students should observe that choice of the asset pricing model appears to make little difference in the results. Hence, the analysis is robust to choice of the asset pricing model. Alert students should object that we have only looked at two of many possible models, and their objection is reasonable. The instructor should encourage the class to look at results in the literature to see what more comprehensive surveys show. For example, students might be asked to read Bennett and Sias (2011), who examine several asset pricing models (including the two in this project). These authors conclude that specification of the asset pricing model has, at best, a minor effect on the estimates of unsystematic shocks, *i.e.*, those due to diversifiable risk.

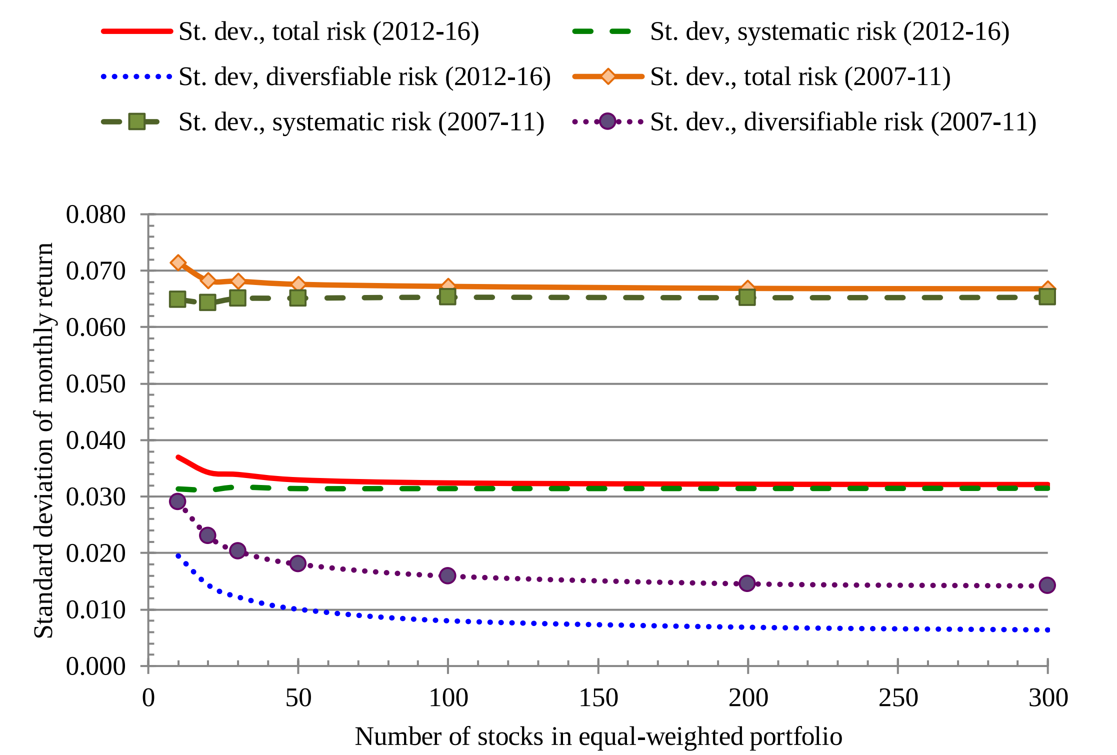
Choice of Historical Period

The last robustness test is somewhat different. In the case of robustness to the specification of the asset pricing model, a natural null hypothesis would be that results should be the same regardless of the model. When comparing results for different historical periods, however, our expectations are somewhat different. The class discussion prior to this analysis should cover two important points. First, based on what we know about diversification, the class might hypothesize that the general patterns will still be present, *i.e.*, naïve diversification still results in a decline in total portfolio risk and convergence to the portfolio's systematic risk as number of stocks in the portfolio increases, and portfolio diversifiable risk also will decline. The class might also hypothesize that a significant amount of diversifiable risk will still be present, even when the portfolio has a large number of stocks.

However, when we compare the effects of diversification in different time periods, students should recognize that economic conditions might affect the effectiveness of the diversification. For example, we could look at two periods that cover different market conditions: 2012-2016, which included a bull market; and 2007-2011, which included a major bear market. The instructor should encourage the class to develop a hypothesis about the relative effectiveness of diversification in each environment. For example, would diversification be more or less helpful in risk reduction when investors need it the most, *i.e.*, in a market downturn when market volatility is greater?

Figure 8 illustrates the type of chart that the instructor might ask students to construct. It shows the medians of the cross-sectional distributions of total, systematic, and diversifiable risks as number of stocks in the portfolio increases. One result (which should not be surprising) is that the general pattern of the medians of the three types of portfolio risk is the same for both periods. Moreover, as the class may have hypothesized, all three types of risk are greater in the period that included a bear market than in the period that included a bull market. In particular, the median diversifiable risk is about twice as large in 2007-2011 than in 2012-2016 at every portfolio size. In other words, naïve diversification was beneficial in both periods but less effective in an absolute sense during the bear market.

Figure 8. Medians: Total, Systematic, and Diversifiable Risk (1-factor market model; 2007-2011 versus 2012-2016 data)



A table that quantifies the dispersion may also be helpful. As an example, see Table 2, which compares the dispersion in terms of the 20th and 80th percentiles at each portfolio size. Students also can construct corresponding charts. Figure 9 illustrates that we often can get a quicker grasp of a key point from a chart than a table of numbers. For example, glancing down Table 2, it is not easy to see that the cross-sectional ***dispersion*** of diversifiable risk is significantly greater for portfolios in 2007-2011 than in 2012-2016. This point is easy to observe in Figure 9.

Table 2. Comparison of Median and Selected Percentiles for Cross-sectional Distributions of Risk Measures for Different Historical Periods

Panel A: Portfolio Systematic Risk (standard deviation of monthly portfolio market return)						
Portfolio size N	Original samples of portfolios over 2012-2016 (which includes a bull market)			Original samples of portfolios over 2007-2011 (which includes a bear market)		
	20th percentile	Median	80th percentile	20th percentile	Median	80th percentile
10	0.027	0.031	0.035	0.056	0.065	0.074
20	0.029	0.031	0.034	0.059	0.064	0.070
30	0.029	0.032	0.034	0.060	0.065	0.070
50	0.030	0.031	0.033	0.062	0.065	0.069
100	0.030	0.031	0.032	0.063	0.065	0.068
200	0.031	0.031	0.032	0.064	0.065	0.067
300	0.031	0.032	0.032	0.064	0.065	0.066
Panel B: Portfolio Diversifiable Risk (standard deviation of monthly portfolio error term)						
Portfolio size N	Original samples of portfolios over 2012-2016 (which includes a bull market)			Original samples of portfolios over 2007-2011 (which includes a bear market)		
	20th percentile	Median	80th percentile	20th percentile	Median	80th percentile
10	0.017	0.020	0.022	0.024	0.029	0.038
20	0.013	0.014	0.016	0.019	0.023	0.028
30	0.011	0.012	0.014	0.017	0.020	0.024
50	0.009	0.010	0.011	0.015	0.018	0.021
100	0.007	0.008	0.009	0.014	0.016	0.018
200	0.006	0.007	0.007	0.013	0.014	0.016
300	0.006	0.006	0.007	0.013	0.014	0.015

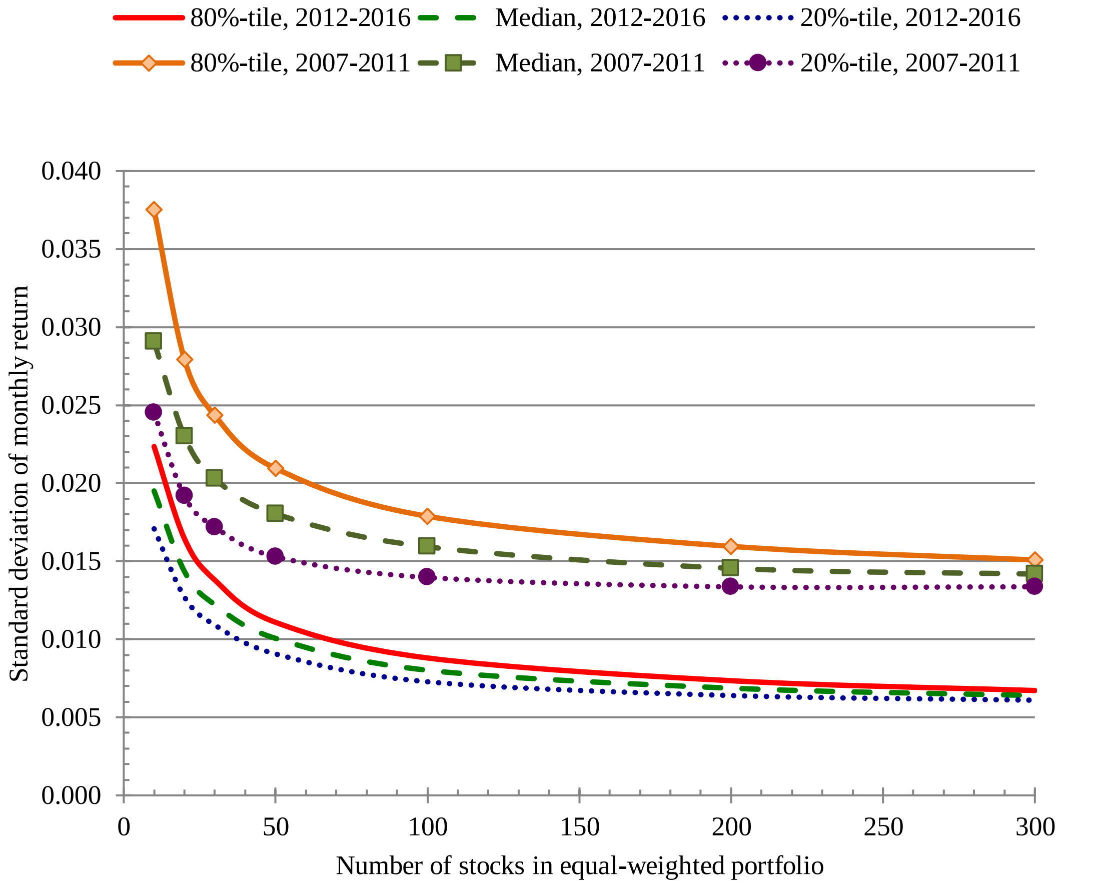
Notes. Cross-sectional statistics are calculated from 1,000 portfolios drawn at random from all possible equal-weighted portfolios at the given portfolio size, where stocks are selected from a list of 498 large-cap U.S. stocks. For both periods, systematic risk (*i.e.*, market risk) and diversifiable risk (*i.e.*, that due to company-unique risks in the portfolio) are estimated from the single-index market model in which the market index is the excess total market return from the Kenneth R. French Data Library (2018). In this table, we use the same samples of portfolios at each portfolio size for both periods.

Conclusions

This class project illustrates that students can perform a good first pass robustness analysis with readily available tools (*e.g.*, Microsoft Excel) and free data. Hence, not having expensive research databases and software is no excuse for failing to examine whether reported results of a study are sensitive to different assumptions and scenarios that might lead to different conclusions. For example, this project illustrates for students in investments courses that results in naïve diversification are sensitive to choice of the market proxy, which is consistent with conclusions in the academic literature. Also, the magnitude of diversifiable shocks and hence the effectiveness of diversification appear to be strongly dependent on the choice of historical period for the analysis, especially when the periods cover different market events such as bear and bull markets.

The project also helps students to see that a good robustness analysis can guide us in deciding what procedures we need to apply. For example, in the case of naïve diversification, the project illustrates that results are robust to sample sizes of 1,000 portfolios at each portfolio size. Hence, we can conduct the analysis with a relatively basic tool such as Excel rather than having to use special software needed for larger sample sizes. Also, based on our limited analysis of two major asset pricing models, the conclusions do not appear to be sensitive to the addition of additional factors beyond a broad stock market index. Thus, we can achieve satisfactory results in this case with a relatively simple model (the SIMM) and do not need to expend time and resources exploring more complicated asset pricing models.

Figure 9. Dispersion of Diversifiable Risk (1-factor market model; 2007-2011 versus 2012-2016 data)



The project helps students to understand that robustness is not an all-or-nothing proposition. For example, in the case of naïve diversification, the students should see the same ***general*** patterns over and over as they perform the robustness analysis. Median total portfolio risk always declines quickly as N, the number of stocks, increases, and eventually levels out close to the market risk. Median diversifiable risk also declines quickly for small values of N, but then at a decreasing rate. On the other hand, it may not converge to zero quickly. Thus, significant diversifiable risk remains even at relatively large portfolio sizes. Dispersion of each type of risk is substantial for smaller portfolio sizes and is not zero even for larger portfolio sizes, although it does decrease significantly.

An ancillary benefit of this particular project is that it may motivate students to engage in critical thinking as they study investments and finance in general. The widespread conventional wisdom that only a small number of stocks is necessary for effective diversification is not true. The robustness analysis in the proposed class project leads to more careful consideration of both dispersion of cross-sectional risks and magnitude of shocks due to remaining diversifiable risk. Thus, the analysis confirms the more recent observation by Bennett and Sias (2011) that what matters is not how close the (average) total risk is to the market risk; what matters is how much diversifiable risk remains.

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